Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(10) The Decomposition of a Polytope into Its Cones

Yoshio Okamoto

Tokyo Institute of Technology

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1. The Identity "$\sum_{m \in \mathbb{Z}} z^m = 0$" ...or "Much Ado About Nothing"

2. Tangent Cones and Their Rational Generating Functions

3. Brion's Theorem

4. Brion Implies Ehrhart

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the denominator of $\mathcal{P}$

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

Conclusion from the previous chapter

We saw the following can be computed efficiently (by means of reciprocity)

- Ehrhart polynomials of the Mordell–Pommersheim tetrahedra
- Ehrhart quasipolynomials of rational convex polygons

Question from the previous chapter

Can we compute the Ehrhart quasipolynomial of any convex polytope efficiently?

Goal of this chapter

Look at mathematical (geometric) ideas that form a basis of efficient algorithm for the task above

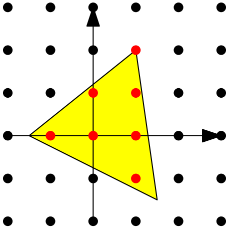
Integer-point transforms

Definition (Integer-point transform, recap)

The integer-point transform of $S \subseteq \mathbb{R}^d$ is

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \ldots, z_d) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$$

Recall: $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$



Example:

$$\sigma_S(z_1, z_2) = z_1 z_2^2 + z_1 z_2 + z_1 + z_1 z_2^{-1} + z_2 + 1 + z_1^{-1}$$

A one-dimensional example (1)

- Consider the line segment $\mathcal{I} := [20, 34]$
- Then

$$\sigma_{\mathcal{I}}(z) = z^{20} + z^{21} + \cdots + z^{34}$$

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- Observation: Long polynomial representation vs. Short rational representation

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- Observation: Long polynomial representation vs. Short rational representation
- We rewrite the expression

$$\sigma_{\mathcal{I}}(z) = \frac{z^{20}}{1 - z} + \frac{z^{34}}{1 - \frac{1}{z}}$$

A one-dimensional example (2)

- As a matter of fact

$$\sigma_{[20,\infty)}(z) = \sum_{m \geq 20} z^m = \frac{z^{20}}{1-z},$$

$$\sigma_{(-\infty,34]}(z) = \sum_{m \leq 34} z^m = \frac{z^{34}}{1-\frac{1}{z}}$$

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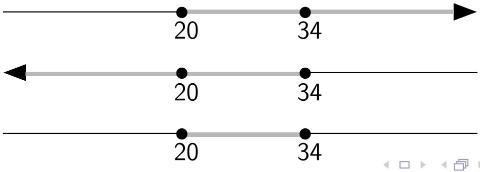
- Therefore, it holds that as rational functions

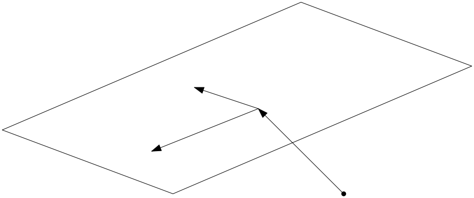
$$\sigma_{[20,\infty)}(z) + \sigma_{(-\infty,34]}(z) = \sigma_{[20,34]}(z)$$



Affine spaces

- Any affine space $\mathcal{A} \subseteq \mathbb{R}^d$ equals $\mathbf{w} + \mathcal{V}$ for some $\mathbf{w} \in \mathbb{R}^d$ and some n-dimensional vector subspace $\mathcal{V} \subseteq \mathbb{R}^d$
- $\mathcal{A}$ contains integer points $\Rightarrow$ we may choose $\mathbf{w} \in \mathbb{Z}^d$
- $\exists$ a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for $\mathcal{V} \cap \mathbb{Z}^d$
- $\therefore$ Any integer point $\mathbf{m} \in \mathcal{A} \cap \mathbb{Z}^d$ can be uniquely written as

$$\mathbf{m} = \mathbf{w} + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n \text{ for some } k_1, k_2, \ldots, k_n \in \mathbb{Z}$$



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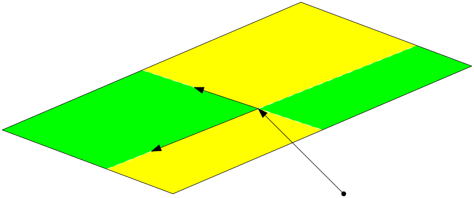

Skewed orthants

Definition (skewed orthant)

Using this fixed lattice basis for $\mathcal{V}$, we define the skewed orthants of $\mathcal{A}$ as the sets of the form

$$\{\mathbf{w} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n\},$$

where for each $1 \le j \le n$, we require either $\lambda_j \ge 0$ or $\lambda_j < 0$

- So there are 2^n such skewed orthants, and their disjoint union equals $\mathcal{A}$
- We denote them by $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{2^n}$
- All of them are (half-open) pointed cones, and so their integer-point transforms are rational

Much Ado about Nothing

Lemma 9.1

Suppose $\mathcal{A}$ is an n-dimensional affine space with skewed orthants $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{2^n}$. Then as rational functions,

$$\sigma_{\mathcal{O}_1}(\mathbf{z}) + \sigma_{\mathcal{O}_2}(\mathbf{z}) + \cdots + \sigma_{\mathcal{O}_{2^n}}(\mathbf{z}) = 0$$

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Proof:

- Suppose

$$\mathcal{A} = \{\mathbf{w} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n : \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}\}$$

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- Then a typical skewed orthant $\mathcal{O}$ looks like

$$\mathcal{O} = \left\{\mathbf{w} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n : \begin{array}{l} \lambda_1, \ldots, \lambda_k \geq 0, \\ \lambda_{k+1}, \ldots, \lambda_n < 0 \end{array}\right\}$$

Proof of Lemma 9.1 (cont'd)

- The integer-point transform of $\mathcal{O}$ is

$$\sigma_{\mathcal{O}}(\mathbf{z}) = \mathbf{z}^{\mathbf{w}} \left(\sum_{j_1 \geq 0} \mathbf{z}^{j_1 \mathbf{v}_1} \right) \cdots \left(\sum_{j_k \geq 0} \mathbf{z}^{j_k \mathbf{v}_k} \right) \left(\sum_{j_{k+1} < 0} \mathbf{z}^{j_{k+1} \mathbf{v}_{k+1}} \right) \cdots \left(\sum_{j_n < 0} \mathbf{z}^{j_n \mathbf{v}_n} \right)$$

$$= \mathbf{z}^{\mathbf{w}} \frac{1}{1 - \mathbf{z}^{\mathbf{v}_1}} \cdots \frac{1}{1 - \mathbf{z}^{\mathbf{v}_k}} \frac{1}{\mathbf{z}^{\mathbf{v}_{k+1}} - 1} \cdots \frac{1}{\mathbf{z}^{\mathbf{v}_n} - 1}$$

Proof of Lemma 9.1 (cont'd)

- The integer-point transform of $\mathcal{O}$ is

$$
\begin{aligned}
&\sigma_{\mathcal{O}}(\mathbf{z}) \\
&= \mathbf{z}^{\mathbf{w}} \left(\sum_{j_1 \geq 0} \mathbf{z}^{j_1\mathbf{v}_1}\right) \cdots \left(\sum_{j_k \geq 0} \mathbf{z}^{j_k\mathbf{v}_k}\right) \left(\sum_{j_{k+1} < 0} \mathbf{z}^{j_{k+1}\mathbf{v}_{k+1}}\right) \cdots \left(\sum_{j_n < 0} \mathbf{z}^{j_n\mathbf{v}_n}\right) \\
&= \mathbf{z}^{\mathbf{w}} \frac{1}{1-\mathbf{z}^{\mathbf{v}_1}} \cdots \frac{1}{1-\mathbf{z}^{\mathbf{v}_k}} \frac{1}{\mathbf{z}^{\mathbf{v}_{k+1}}-1} \cdots \frac{1}{\mathbf{z}^{\mathbf{v}_n}-1}
\end{aligned}
$$

- Consider the skewed orthant $\mathcal{O}'$ with the same conditions on the λ's as in $\mathcal{O}$ except that we switch $\lambda_1 \geq 0$ to $\lambda_1 < 0$

Proof of Lemma 9.1 (cont'd)

- The integer-point transform of $\mathcal{O}$ is

$$\sigma_{\mathcal{O}}(\mathbf{z}) = \mathbf{z}^{\mathbf{w}} \left(\sum_{j_1 \geq 0} \mathbf{z}^{j_1 \mathbf{v}_1} \right) \cdots \left(\sum_{j_k \geq 0} \mathbf{z}^{j_k \mathbf{v}_k} \right) \left(\sum_{j_{k+1} < 0} \mathbf{z}^{j_{k+1} \mathbf{v}_{k+1}} \right) \cdots \left(\sum_{j_n < 0} \mathbf{z}^{j_n \mathbf{v}_n} \right)$$

$$= \mathbf{z}^{\mathbf{w}} \frac{1}{1 - \mathbf{z}^{\mathbf{v}_1}} \cdots \frac{1}{1 - \mathbf{z}^{\mathbf{v}_k}} \frac{1}{\mathbf{z}^{\mathbf{v}_{k+1}} - 1} \cdots \frac{1}{\mathbf{z}^{\mathbf{v}_n} - 1}$$

- Consider the skewed orthant $\mathcal{O}'$ with the same conditions on the λ's as in $\mathcal{O}$ except that we switch $\lambda_1 \geq 0$ to $\lambda_1 < 0$; Then the integer-point transform of $\mathcal{O}'$ is

$$\sigma_{\mathcal{O}'}(\mathbf{z}) = \mathbf{z}^{\mathbf{w}} \frac{1}{\mathbf{z}^{\mathbf{v}_1} - 1} \frac{1}{1 - \mathbf{z}^{\mathbf{v}_2}} \cdots \frac{1}{1 - \mathbf{z}^{\mathbf{v}_k}} \frac{1}{\mathbf{z}^{\mathbf{v}_{k+1}} - 1} \cdots \frac{1}{\mathbf{z}^{\mathbf{v}_n} - 1}$$

Proof of Lemma 9.1 (further cont'd)

- $\therefore \sigma_{\mathcal{O}}(\mathbf{z}) + \sigma_{\mathcal{O}'}(\mathbf{z}) = 0$

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- $\therefore \sigma_{\mathcal{O}}(\mathbf{z}) + \sigma_{\mathcal{O}'}(\mathbf{z}) = 0$
- Since we can pair up all skewed orthants in this fashion, the sum of all their rational generating functions is zero □

Proof of Lemma 9.1 (further cont'd)

- $\therefore \sigma_{\mathcal{O}}(\mathbf{z}) + \sigma_{\mathcal{O}'}(\mathbf{z}) = 0$
- Since we can pair up all skewed orthants in this fashion, the sum of all their rational generating functions is zero □

Since $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_{2^n}$ is equal to $\mathcal{A}$ as a disjoint union, it now makes sense to set

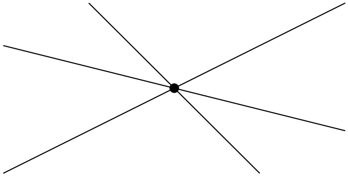
$$\sigma_{\mathcal{A}}(\mathbf{z}) := 0$$

when $\mathcal{A}$ is an affine space of dimension $n > 0$

Hyperplane arrangements

Definition (Hyperplane arrangement)

- A hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes
- An arrangement $\mathcal{H}$ is rational if all its hyperplanes are, that is, if each hyperplane in $\mathcal{H}$ is of the form $\left\{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d = b\right\}$ for some $a_1, a_2, \ldots, a_d, b \in \mathbb{Z}$
- An arrangement $\mathcal{H}$ is called a central hyperplane arrangement if its hyperplanes meet in (at least) one point



Convex cones

Definition (Convex cone)

A convex cone is the intersection of finitely many half-spaces of the form $\{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d \leq b\}$ for which the corresponding hyperplanes $\{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d = b\}$ form a central arrangement

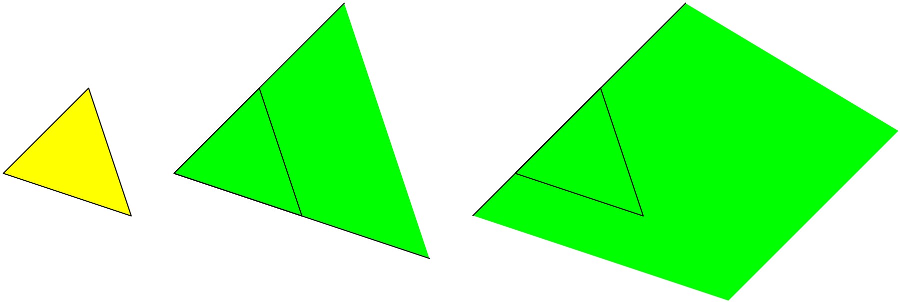
- This definition extends that of a pointed cone: a cone is pointed if the defining hyperplanes meet in *exactly* one point
- A cone is rational if all of its defining hyperplanes are rational
- Cones and polytopes are special cases of polyhedra, which are convex bodies defined as the intersection of finitely many half-spaces

Tangent cones

Definition (Tangent cone)

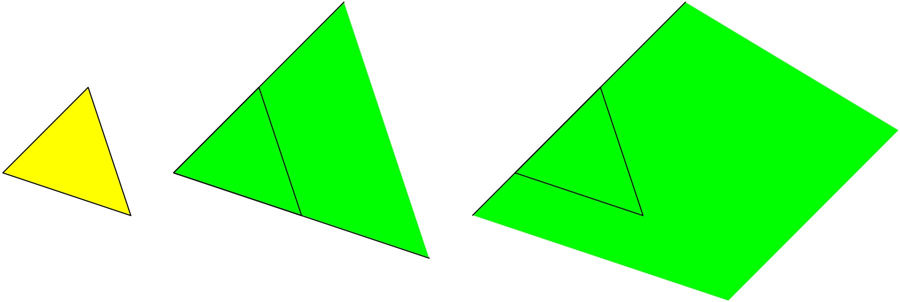
For a face $\mathcal{F}$ of a convex poyltope $\mathcal{P}$, define its tangent cone as

$$\mathcal{K}_{\mathcal{F}} := \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x} \in \mathcal{F}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}_{\geq 0}\}$$



Properties of tangent cones

- $\mathcal{K}_{\mathcal{F}}$ is the smallest convex cone containing both span $\mathcal{F}$ and $\mathcal{P}$
- $\mathcal{K}_{\mathcal{P}} = \operatorname{span} \mathcal{P}$
- $\mathcal{K}_{\mathbf{v}}$ is often called a vertex cone if $\mathbf{v}$ is a vertex of $\mathcal{P}$; it is pointed
- $\mathcal{K}_{\mathcal{F}}$ is not pointed for a k-face $\mathcal{F}$ of $\mathcal{P}$ with $k > 0$



Tangent cones and the spans of faces

Lemma 9.3

For any face $\mathcal{F}$ of $\mathcal{P}$, $\operatorname{span}\mathcal{F} \subseteq \mathcal{K}_{\mathcal{F}}$.

Proof: As $\mathbf{x}$ and $\mathbf{y}$ vary over all points of $\mathcal{F}$, $\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$ varies over $\operatorname{span}\mathcal{F}$ □

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Remarks

- Lem 9.3 implies that $\mathcal{K}_{\mathcal{F}}$ contains a line, unless $\mathcal{F}$ is a vertex
- More precisely, if $\mathcal{K}_{\mathcal{F}}$ is not pointed, it contains the affine space $\operatorname{span} \mathcal{F}$, which is called the apex of the tangent cone
- (A pointed cone has a point as apex)

Orthogonal complements of affine spaces

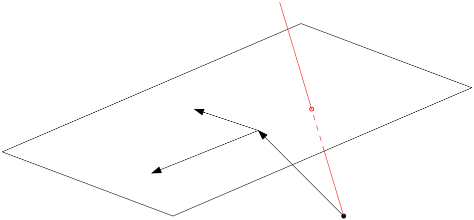
Reminder: An affine space $\mathcal{A} \subseteq \mathbb{R}^d$ equals $\mathbf{w} + \mathcal{V}$ for some $\mathbf{w} \in \mathbb{R}^d$ and some vector subspace $\mathcal{V} \subseteq \mathbb{R}^d$

Definition (Orthogonal complement)

The orthogonal complement $\mathcal{A}^\perp$ of this affine space $\mathcal{A}$ is defined by

$$\mathcal{A}^\perp := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in \mathcal{V}\}$$

Note: $\mathcal{A} \oplus \mathcal{A}^\perp = \mathbb{R}^d$



A decomposition of tangent cones

Lemma 9.4

For any face $\mathcal{F}$ of $\mathcal{P}$, the tangent cone $\mathcal{K}_{\mathcal{F}}$ has the decomposition

$$\mathcal{K}_{\mathcal{F}} = \operatorname{span} \mathcal{F} \oplus \left((\operatorname{span} \mathcal{F})^{\perp} \cap \mathcal{K}_{\mathcal{F}} \right);$$

Consequently, $\sigma_{\mathcal{K}_{\mathcal{F}}}(\mathbf{z}) = 0$ unless $\mathcal{F}$ is a vertex

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Proof of the 1st part:

- Since $\operatorname{span} \mathcal{F} \oplus (\operatorname{span} \mathcal{F})^{\perp} = \mathbb{R}^d$,

$$\mathcal{K}_{\mathcal{F}} = \left(\operatorname{span} \mathcal{F} \oplus (\operatorname{span} \mathcal{F})^{\perp} \right) \cap \mathcal{K}_{\mathcal{F}}$$

(Lem 9.3)

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Proof of the 1st part:

- Since $\operatorname{span}\mathcal{F} \oplus (\operatorname{span}\mathcal{F})^\perp = \mathbb{R}^d$,

$$\begin{aligned}\mathcal{K}_\mathcal{F} &= \left(\operatorname{span}\mathcal{F} \oplus (\operatorname{span}\mathcal{F})^\perp\right) \cap \mathcal{K}_\mathcal{F} \\ &= (\operatorname{span}\mathcal{F} \cap \mathcal{K}_\mathcal{F}) \oplus \left((\operatorname{span}\mathcal{F})^\perp \cap \mathcal{K}_\mathcal{F} \right) \\ &= \operatorname{span}\mathcal{F} \oplus \left((\operatorname{span}\mathcal{F})^\perp \cap \mathcal{K}_\mathcal{F} \right) \qquad \text{(Lem 9.3)}\end{aligned}$$

A decomposition of tangent cones: 2nd part

Lemma 9.4

For any face $\mathcal{F}$ of $\mathcal{P}$, the tangent cone $\mathcal{K}_{\mathcal{F}}$ has the decomposition

$$\mathcal{K}_{\mathcal{F}} = \operatorname{span}\mathcal{F} \oplus \left((\operatorname{span}\mathcal{F})^{\perp} \cap \mathcal{K}_{\mathcal{F}}\right);$$

Consequently, $\sigma_{\mathcal{K}_{\mathcal{F}}}(\mathbf{z}) = 0$ unless $\mathcal{F}$ is a vertex

Proof of the 2nd part:

- Immediate from the 1st part since
  - $\sigma_{\operatorname{span}\mathcal{F} \oplus \left((\operatorname{span}\mathcal{F})^{\perp} \cap \mathcal{K}_{\mathcal{F}}\right)}(\mathbf{z}) = \sigma_{\operatorname{span}\mathcal{F}}(\mathbf{z})\, \sigma_{(\operatorname{span}\mathcal{F})^{\perp} \cap \mathcal{K}_{\mathcal{F}}}(\mathbf{z})$ and
  - $\sigma_{\operatorname{span}\mathcal{F}}(\mathbf{z}) = 0$ $\square$

Brion's theorem

The main theorem of this chapter

Theorem 9.7 (Brion's theorem)

Suppose $\mathcal{P}$ is a rational convex polytope. Then as rational functions:

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z})$$

Roadmap for the proof:

- Prove Thm 9.5 (Brianchon–Gram identity for simplices)
- Prove Cor 9.6 (Brion's theorem for simplices)
- Prove Thm 9.7

Brianchon–Gram identity for simplices

Definition (Indicator function)

The indicator function 1_S of a set $S \subset \mathbb{R}^d$ is defined by

$$1_S(\mathbf{x}) := \begin{cases} 1 & \text{if} \quad \mathbf{x} \in S, \\ 0 & \text{if} \quad \mathbf{x} \notin S \end{cases}$$

Brianchon–Gram identity for simplices

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Theorem 9.5 (Brianchon–Gram identity for simplices)

Let Δ be a d-simplex. Then

$$1_\Delta(\mathbf{x}) = \sum_{\mathcal{F} \subseteq \Delta} (-1)^{\dim \mathcal{F}} 1_{\mathcal{K}_\mathcal{F}}(\mathbf{x}),$$

where the sum is taken over all nonempty faces $\mathcal{F}$ of Δ

Proof of Brianchon–Gram for simplicies

We distinguish between two disjoint cases: whether or not $\mathbf{x}$ is in Δ

- Case 1: $\mathbf{x} \in \Delta$
- Case 2: $\mathbf{x} \notin \Delta$

□

Proof of Brianchon–Gram for simplicies

We distinguish between two disjoint cases: whether or not $\mathbf{x}$ is in Δ

- Case 1: $\mathbf{x} \in \Delta$
  - $\mathbf{x} \in \mathcal{K}_{\mathcal{F}}$ for all $\mathcal{F} \subseteq \Delta$
- Case 2: $\mathbf{x} \notin \Delta$

□

Proof of Brianchon–Gram for simplicies

We distinguish between two disjoint cases: whether or not $\mathbf{x}$ is in Δ

- Case 1: $\mathbf{x} \in \Delta$
  - $\mathbf{x} \in \mathcal{K}_{\mathcal{F}}$ for all $\mathcal{F} \subseteq \Delta$
  - Then

$$1 = \sum_{k=0}^{\dim \Delta} (-1)^k f_k$$

    by the Euler relation for simplices (Exer 5.5)

- Case 2: $\mathbf{x} \notin \Delta$

□

Proof of Brianchon–Gram for simplicies

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  - $\mathbf{x} \in \mathcal{K}_{\mathcal{F}}$ for all $\mathcal{F} \subseteq \Delta$
  - Then
$$1 = \sum_{k=0}^{\dim \Delta} (-1)^k f_k = \sum_{\mathcal{F} \subseteq \Delta} (-1)^{\dim \mathcal{F}}$$
    by the Euler relation for simplices (Exer 5.5)
- Case 2: $\mathbf{x} \notin \Delta$

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Proof of Brianchon–Gram for simplicies

We distinguish between two disjoint cases: whether or not $\mathbf{x}$ is in Δ

- Case 1: $\mathbf{x} \in \Delta$
  - $\mathbf{x} \in \mathcal{K}_{\mathcal{F}}$ for all $\mathcal{F} \subseteq \Delta$
  - Then
$$1 = \sum_{k=0}^{\dim \Delta} (-1)^k f_k = \sum_{\mathcal{F} \subseteq \Delta} (-1)^{\dim \mathcal{F}}$$
    by the Euler relation for simplices (Exer 5.5)
- Case 2: $\mathbf{x} \notin \Delta$
  - $\exists !$ a minimal face $\mathcal{F} \subseteq \Delta$ (w.r.t. dimension) s.t. $\mathbf{x} \in \mathcal{K}_{\mathcal{F}}$ and $\mathbf{x} \in \mathcal{K}_{\mathcal{G}}$ for all faces $\mathcal{G} \subseteq \Delta$ that contain $\mathcal{F}$ (Exer 9.2)

□

Proof of Brianchon–Gram for simplicies

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  - Then $0 = \sum_{\mathcal{G} \supseteq \mathcal{F}} (-1)^{\dim \mathcal{G}}$ (Exer 9.4)

□

Brion's theorem for simplices

Corollary 9.6 (Brion's theorem for simplices)

Suppose Δ is a rational simplex. Then as rational functions:

$$\sigma_{\Delta}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \Delta} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z})$$

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Proof:

- We sum both sides of the identity in Thm 9.5 $\forall\, \mathbf{m} \in \mathbb{Z}^d$:

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} 1_\Delta(\mathbf{m})\,\mathbf{z}^\mathbf{m} \;=\; \sum_{\mathbf{m}\in\mathbb{Z}^d} \sum_{\mathcal{F}\subseteq\Delta} (-1)^{\dim \mathcal{F}} 1_{\mathcal{K}_\mathcal{F}}(\mathbf{m})\,\mathbf{z}^\mathbf{m}$$

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$$\begin{aligned} \sum_{\mathbf{m}\in\mathbb{Z}^d} 1_\Delta(\mathbf{m})\,\mathbf{z}^\mathbf{m} &= \sum_{\mathbf{m}\in\mathbb{Z}^d} \sum_{\mathcal{F}\subseteq\Delta} (-1)^{\dim \mathcal{F}} 1_{\mathcal{K}_\mathcal{F}}(\mathbf{m})\,\mathbf{z}^\mathbf{m} \\ \therefore \qquad \sigma_\Delta(\mathbf{z}) &= \sum_{\mathcal{F}\subseteq\Delta} (-1)^{\dim \mathcal{F}} \sigma_{\mathcal{K}_\mathcal{F}}(\mathbf{z}) \end{aligned}$$

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- But Lem 9.4 implies that $\sigma_{\mathcal{K}_\mathcal{F}}(\mathbf{z}) = 0$ unless $\mathcal{F}$ is a vertex; Hence

$$\sigma_\Delta(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \Delta} \sigma_{\mathcal{K}_\mathbf{v}}(\mathbf{z}) \qquad \square$$

Proof of Brion's theorem (1)

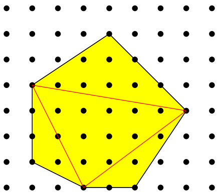
Theorem 9.7 (Brion's theorem)

Suppose $\mathcal{P}$ is a rational convex polytope. Then as rational functions:

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z})$$

Proof: We use the same irrational trick as in the proofs of Thms 3.12 & 4.3

- Triangulate $\mathcal{P}$ into the simplices $\Delta_1, \Delta_2, \ldots, \Delta_m$ (using no new vertices)

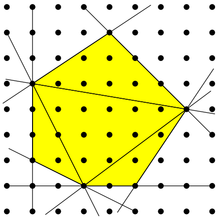


Proof of Brion's theorem (2)

- Consider the hyperplane arrangement

$$\mathcal{H} := \{\mathrm{span}\,\mathcal{F} : \mathcal{F} \text{ is a facet of } \Delta_1, \Delta_2, \ldots, \text{ or } \Delta_m\}$$

- Now shift the hyperplanes in $\mathcal{H}$, obtaining a new hyperplane arrangement $\mathcal{H}^{\text{shift}}$



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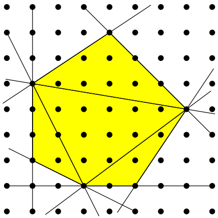
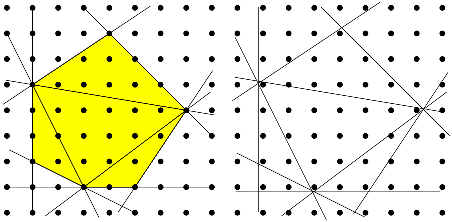
- Now shift the hyperplanes in $\mathcal{H}$, obtaining a new hyperplane arrangement $\mathcal{H}^{\text{shift}}$
- Those hyperplanes of $\mathcal{H}$ that defined $\mathcal{P}$ now define, after shifting, a new polytope that we will call $\mathcal{P}^{\text{shift}}$



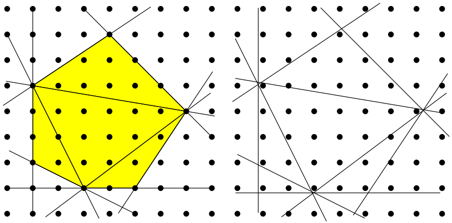
Proof of Brion's theorem (3)

- Exer 9.6 ensures that we can shift $\mathcal{H}$ in such a way that:
  - No hyperplane in $\mathcal{H}^{\text{shift}}$ contains any lattice point
  - $\mathcal{H}^{\text{shift}}$ yields a triangulation of $\mathcal{P}^{\text{shift}}$
  - The lattice points contained in a vertex cone of $\mathcal{P}$ are precisely the lattice points contained in the corresp. vertex cone of $\mathcal{P}^{\text{shift}}$



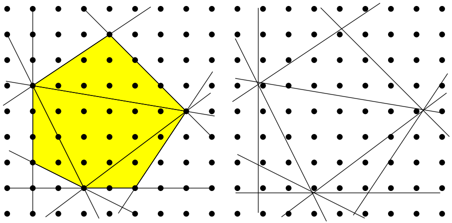
Proof of Brion's theorem (4)

- This setup implies that
  - the lattice points in $\mathcal{P}$ are precisely the lattice points in $\mathcal{P}^{\text{shift}}$
  - the lattice points in a vertex cone of $\mathcal{P}^{\text{shift}}$ can be written as a *disjoint* union of lattice points in vertex cones of simplices of the triangulation that $\mathcal{H}^{\text{shift}}$ induces on $\mathcal{P}^{\text{shift}}$



Proof of Brion's theorem (4)

- This setup implies that
  - the lattice points in $\mathcal{P}$ are precisely the lattice points in $\mathcal{P}^{\text{shift}}$
  - the lattice points in a vertex cone of $\mathcal{P}^{\text{shift}}$ can be written as a *disjoint* union of lattice points in vertex cones of simplices of the triangulation that $\mathcal{H}^{\text{shift}}$ induces on $\mathcal{P}^{\text{shift}}$
- These conditions, in turn, mean that Brion follows from Brion for simplices: the integer-point transforms on both sides of the identity can be written as a sum of integer-point transforms of simplices and their vertex cones □



1. The Identity "$\sum_{m \in \mathbb{Z}} z^m = 0$" ... or "Much Ado About Nothing"
2. Tangent Cones and Their Rational Generating Functions
3. Brion's Theorem
4. Brion Implies Ehrhart

Proof of Ehrhart's theorem for rational polytopes

... by Brion's theorem

- As in our first proof of Ehrhart's thm, it suffices to prove Ehrhart's thm for simplices, because we can triangulate any polytope (using only the vertices)
- So suppose Δ is a rational d-simplex whose vertices have coordinates with denominator p

Proof of Ehrhart's theorem for rational polytopes

... by Brion's theorem

- As in our first proof of Ehrhart's thm, it suffices to prove Ehrhart's thm for simplices, because we can triangulate any polytope (using only the vertices)
- So suppose Δ is a rational d-simplex whose vertices have coordinates with denominator p
- Goal: for a fixed $0 \leq r < p$, the function $L_\Delta(r + pt)$ is a polynomial in t
    - This means that L_Δ is a quasipolynomial with period dividing p

Proof (2)

- Then,

$$L_{\Delta}(r + pt) = \sum_{\mathbf{m} \in (r+pt)\Delta \cap \mathbb{Z}^d} 1$$

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$$L_\Delta(r+pt) = \sum_{\mathbf{m}\in(r+pt)\Delta\cap\mathbb{Z}^d} 1 = \lim_{\mathbf{z}\to\mathbf{1}} \sigma_{(r+pt)\Delta}(\mathbf{z})$$

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$$= \lim_{\mathbf{z}\to\mathbf{1}} \sum_{\mathbf{v}\text{ a vertex of }\Delta} \sigma_{(r+pt)\mathcal{K}_\mathbf{v}}(\mathbf{z})$$

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$$= \lim_{\mathbf{z}\to\mathbf{1}} \sum_{\mathbf{v} \text{ a vertex of } \Delta} \sigma_{(r+pt)\mathcal{K}_\mathbf{v}}(\mathbf{z})$$

- Note: $\mathcal{K}_\mathbf{v}$ are all simplicial, because Δ is a simplex
- So suppose

$$\mathcal{K}_\mathbf{v} = \{\mathbf{v} + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_d\mathbf{w}_d : \lambda_1, \lambda_2, \ldots, \lambda_d \geq 0\}$$

Proof (3)

- Then

$$(r + pt)\mathcal{K}_{\mathbf{v}} = \{(r + pt)\mathbf{v} + \lambda_1\mathbf{w}_1 + \cdots + \lambda_d\mathbf{w}_d : \lambda_1, \ldots, \lambda_d \geq 0\}$$

Proof (3)

- Then

$$
\begin{aligned}
&(r + pt)\mathcal{K}_{\mathbf{v}} \\
&= \{(r + pt)\mathbf{v} + \lambda_1 \mathbf{w}_1 + \cdots + \lambda_d \mathbf{w}_d : \lambda_1, \ldots, \lambda_d \geq 0\} \\
&= tp\mathbf{v} + \{r\mathbf{v} + \lambda_1 \mathbf{w}_1 + \cdots + \lambda_d \mathbf{w}_d : \lambda_1, \ldots, \lambda_d \geq 0\}
\end{aligned}
$$

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&= tp\mathbf{v} + r\mathcal{K}_{\mathbf{v}}
\end{aligned}
$$

Proof (3)

- Then

$$
\begin{aligned}
&(r+pt)\mathcal{K}_{\mathbf{v}}\\
&=\{(r+pt)\mathbf{v}+\lambda_1\mathbf{w}_1+\cdots+\lambda_d\mathbf{w}_d : \lambda_1,\dots,\lambda_d\geq 0\}\\
&=tp\mathbf{v}+\{r\mathbf{v}+\lambda_1\mathbf{w}_1+\cdots+\lambda_d\mathbf{w}_d : \lambda_1,\dots,\lambda_d\geq 0\}\\
&=tp\mathbf{v}+r\mathcal{K}_{\mathbf{v}}
\end{aligned}
$$

- Important to note: $p\mathbf{v}$ is an integer vector
- In particular, we can safely write $\sigma_{(r+pt)\mathcal{K}_{\mathbf{v}}}(\mathbf{z}) = \mathbf{z}^{tp\mathbf{v}}\sigma_{r\mathcal{K}_{\mathbf{v}}}(\mathbf{z})$

Proof (4)

- Now we can rewrite as

$$L_\Delta(r + pt) = \lim_{\mathbf{z}\to\mathbf{1}} \sum_{\mathbf{v} \text{ vertex of } \Delta} \mathbf{z}^{tp\mathbf{v}} \sigma_{r\mathcal{K}_\mathbf{v}}(\mathbf{z})$$

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- The exact forms of the rational functions $\sigma_{r\mathcal{K}_\mathbf{v}}(\mathbf{z})$ is not important, except for the fact that they do not depend on t
- To compute $L_\Delta(r+pt)$, we write all the rational functions on the RHS over one denominator and use L'Hôpital's rule to compute the limit of this one huge rational function

Proof (5)

- We wrote

$$L_\Delta(r+pt) = \lim_{\mathbf{z}\to\mathbf{1}} \sum_{\mathbf{v}\text{ vertex of }\Delta} \mathbf{z}^{tp\mathbf{v}} \sigma_{r\mathcal{K}_\mathbf{v}}(\mathbf{z})$$

- The variable t appears only in the simple monomials $\mathbf{z}^{tp\mathbf{v}}$, so the effect of L'Hôpital's rule is that we obtain linear factors of t every time we differentiate the numerator of this rational function
- At the end we evaluate the remaining rational function at $\mathbf{z} = \mathbf{1}$
- The result is a polynomial in t □