

Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(9) Dedekind Sums

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- ① Fourier–Dedekind Sums and the Coin-Exchange Problem Revisited
- ② The Dedekind Sum and Its Reciprocity and Computational Complexity
- ③ Rademacher Reciprocity for the Fourier–Dedekind Sum
- ④ The Mordell–Pommersheim Tetrahedron

Important theorems from the previous lectures

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d -polytope $\Rightarrow$
 $L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d -polytope $\Rightarrow$
 $L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d ;
Its period divides the denominator of $\mathcal{P}$

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

The goal of this chapter

The goal of this chapter

- ① Look at Fourier–Dedekind sums more closely
- ② In particular, their reciprocity
- ③ Application to their computation

1 Fourier–Dedekind Sums and the Coin-Exchange Problem Revisited

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Reminder: Fourier–Dedekind sums

Definition (Fourier–Dedekind sum)

$$s_n(a_1, a_2, \dots, a_d; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{(1 - \xi_b^{ka_1})(1 - \xi_b^{ka_2}) \cdots (1 - \xi_b^{ka_d})}$$

- Note: $\xi_b = e^{2\pi i/b}$
- Appeared in computation of $p_A(n)$ in Chap 1

Definition (Restricted partition function)

$$p_A(n) = \# \left\{ (m_1, \dots, m_d) \in \mathbb{Z}^d : \begin{array}{l} \text{all } m_j \geq 0, \\ m_1 a_1 + \cdots + m_d a_d = n \end{array} \right\},$$

for $A = \{a_1, \dots, a_d\}$ a set of coprime positive integers

Restricted partition functions (from Chap 1)

For two or three coins

$$\begin{aligned} p_{\{a_1, a_2\}}(n) &= \frac{1}{2a_1} + \frac{1}{2a_2} + \frac{n}{a_1 a_2} + s_{-n}(a_1; a_2) + s_{-n}(a_2; a_1), \\ p_{\{a_1, a_2, a_3\}}(n) &= \frac{n^2}{2a_1 a_2 a_3} + \frac{n}{2} \left(\frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} \right) \\ &\quad + \frac{1}{12} \left(\frac{3}{a_1} + \frac{3}{a_2} + \frac{3}{a_3} + \frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2} \right) \\ &\quad + s_{-n}(a_1, a_2; a_3) + s_{-n}(a_2, a_3; a_1) + s_{-n}(a_1, a_3; a_2) \end{aligned}$$

Example 8.1: When $n = 0$ and $d = 2$

a, b coprime positive integers

$$\begin{aligned} s_0(a, 1; b) &= \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{(1 - \xi_b^{ka})(1 - \xi_b^k)} \\ &= \frac{1}{b} \sum_{k=1}^{b-1} \left(\frac{1}{1 - \xi_b^{ka}} - \frac{1}{2} \right) \left(\frac{1}{1 - \xi_b^k} - \frac{1}{2} \right) \\ &\quad + \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^k} + \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{ka}} - \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{4} \\ &= \underbrace{\frac{1}{4b} \sum_{k=1}^{b-1} \left(\frac{1 + \xi_b^{ka}}{1 - \xi_b^{ka}} \right) \left(\frac{1 + \xi_b^k}{1 - \xi_b^k} \right)}_{-s(a, b)} + \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^k} - \frac{b-1}{4b} \end{aligned}$$

Example 8.1: When $n = 0$ and $d = 2$ (cont'd)

$$\begin{aligned}
s_0(a, 1; b) &= -s(a, b) + \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^k} - \frac{b-1}{4b} \\
&= -s(a, b) + \frac{1}{2} - \frac{1}{2b} - \frac{b-1}{4b} \\
&= -s(a, b) + \frac{b-1}{4b} \quad \square
\end{aligned}$$

Reminder (from Chap 1)

$$\frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{(1 - \xi_b^k) \xi_b^{kn}} = -\left\{ \frac{n}{b} \right\} + \frac{1}{2} - \frac{1}{2b}$$

Example 8.2: When $n = 0$ and $d = 2$

Exercise 8.5

For a_1, a_2 coprime to b ,

$$s_0(a_1, a_2; b) = -s(a_1 a_2^{-1}, b) + \frac{b-1}{4b},$$

where $a_2^{-1} a_2 \equiv 1 \pmod{b}$

How the Fourier–Dedekind sums arose in Chap 1

 $a_1, \dots, a_d$ pairwise coprime

$$\begin{aligned}
f(z) &= \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_d}) z^n} \\
&= \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} + \frac{B_1}{z-1} + \frac{B_2}{(z-1)^2} + \cdots + \frac{B_d}{(z-1)^d} \\
&\quad + \sum_{k=1}^{a_1-1} \frac{C_{1k}}{z - \xi_{a_1}^k} + \sum_{k=1}^{a_2-1} \frac{C_{2k}}{z - \xi_{a_2}^k} + \cdots + \sum_{k=1}^{a_d-1} \frac{C_{dk}}{z - \xi_{a_d}^k} \\
p_A(n) &= \text{const } f(z) \\
&= -B_1 + B_2 - \cdots + (-1)^d B_d + s_{-n}(a_2, a_3, \dots, a_d; a_1) \\
&\quad + s_{-n}(a_1, a_3, a_4, \dots, a_d; a_2) + \cdots + s_{-n}(a_1, a_2, \dots, a_{d-1}; a_d)
\end{aligned}$$

Note: $B_1, \dots, B_d$ are polynomials in n (Exer, 8.6)The polynomial part of $p_A(n)$

Definition (Polynomial part)

The **polynomial part** of $p_A(n)$ is

$$\text{poly}_A(n) := -B_1 + B_2 - \cdots + (-1)^d B_d$$

Example: $n = 2$

$$p_{\{a_1, a_2\}}(n) = \frac{1}{2a_1} + \frac{1}{2a_2} + \frac{n}{a_1 a_2} + s_{-n}(a_1; a_2) + s_{-n}(a_2; a_1)$$

Therefore

$$\text{poly}_{\{a_1, a_2\}}(n) = \frac{1}{2a_1} + \frac{1}{2a_2} + \frac{n}{a_1 a_2}$$

Zagier's reciprocity

Theorem 8.4 (Zagier's reciprocity)

For any pairwise coprime positive integers $a_1, a_2, \dots, a_d$,

$$\begin{aligned} s_0(a_2, a_3, \dots, a_d; a_1) + s_0(a_1, a_3, a_4, \dots, a_d; a_2) + \dots \\ + s_0(a_1, a_2, \dots, a_{d-1}; a_d) \\ = 1 - \text{poly}_{\{a_1, a_2, \dots, a_d\}}(0) \end{aligned}$$

Example: $d = 2$

$$s_0(a_1; a_2) + s_0(a_2; a_1) = 1 - \frac{1}{2a_1} - \frac{1}{2a_2}$$

Proof of Zagier's reciprocity

- By computing the constant term of $p_A(n)$

$$\begin{aligned} p_A(0) &= \text{poly}_A(0) \\ &\quad + s_0(a_2, a_3, \dots, a_d; a_1) \\ &\quad + s_0(a_1, a_3, a_4, \dots, a_d; a_2) \\ &\quad + \dots + s_0(a_1, a_2, \dots, a_{d-1}; a_d) \end{aligned}$$

- $p_A(0) = 1$ (Exer 3.27)

- $\therefore$

$$\begin{aligned} 1 &= \text{poly}_A(0) \\ &\quad + s_0(a_2, a_3, \dots, a_d; a_1) \\ &\quad + s_0(a_1, a_3, a_4, \dots, a_d; a_2) \\ &\quad + \dots + s_0(a_1, a_2, \dots, a_{d-1}; a_d) \end{aligned}$$

□

The Dedekind Sum and Its Reciprocity

The Dedekind Sum and Its Reciprocity

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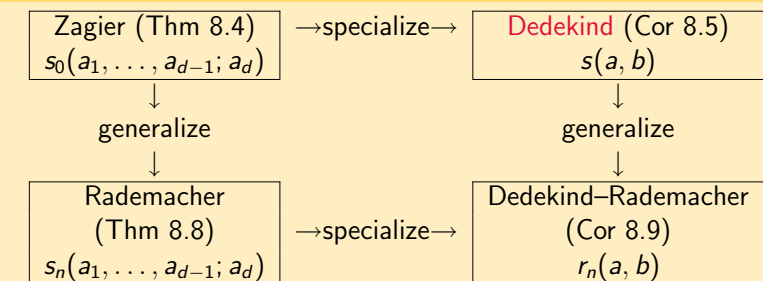
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Relations among reciprocity theorems



Reciprocity of the Dedekind sums

Corollary 8.5 (Dedekind's reciprocity law)

For any coprime positive integers a and b ,

$$s(a, b) + s(b, a) = \frac{1}{12} \left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right) - \frac{1}{4}$$

Proof:

- $s_0(a, 1; b) = -s(a, b) + \frac{b-1}{4b}$ (Example 8.1)
- $s_0(b, a; 1) = 0$ (by Def)
- With Zagier's reciprocity

$$\begin{aligned} s_0(a, 1; b) + s_0(b, a; 1) + s_0(1, b; a) \\ = 1 - \frac{1}{12} \left(\frac{3}{a} + 3 + \frac{3}{b} + \frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right) \quad \square \end{aligned}$$

Computational aspect

A consequence of Dedekind's reciprocity law

It enables us to compute Dedekind sums efficiently

- Reminder: $s(a, b) = s(a \bmod b, b)$

Example 8.6: $a = 100, b = 147$

$$\begin{aligned} s(100, 147) &= \frac{1}{12} \left(\frac{100}{147} + \frac{147}{100} + \frac{1}{14700} \right) - \frac{1}{4} - s(147, 100) \\ &= -\frac{1249}{17640} - s(47, 100) \\ &= -\frac{1249}{17640} - \left(\frac{1}{12} \left(\frac{47}{100} + \frac{100}{47} + \frac{1}{4700} \right) - \frac{1}{4} - s(100, 47) \right) \\ &= -\frac{773}{20727} + s(6, 47) \end{aligned}$$

Computational aspect (cont'd)

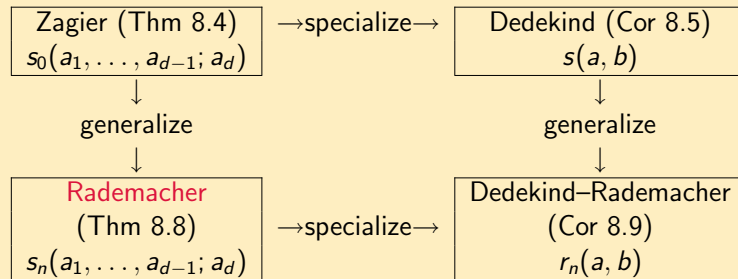
$$\begin{aligned} s(100, 147) &= -\frac{773}{20727} + \frac{1}{12} \left(\frac{6}{47} + \frac{47}{6} + \frac{1}{282} \right) - \frac{1}{4} - s(47, 6) \\ &= \frac{166}{441} - s(5, 6) \\ &= \frac{166}{441} - \left(\frac{1}{12} \left(\frac{5}{6} + \frac{6}{5} + \frac{1}{30} \right) - \frac{1}{4} - s(6, 5) \right) \\ &= \frac{2003}{4410} + s(1, 5) \\ &= \frac{2003}{4410} - \frac{1}{4} + \frac{1}{30} + \frac{5}{12} \\ &= \frac{577}{882} \end{aligned}$$

where we used $s(1, k) = -\frac{1}{4} + \frac{1}{6k} + \frac{k}{12}$ (Exer 7.20)

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Relations among reciprocity theorems



Rademacher reciprocity

Theorem 8.8 (Rademacher reciprocity)

$a_1, a_2, \dots, a_d$ pairwise coprime positive integers
 $n \in \{1, 2, \dots, a_1 + \dots + a_d - 1\} \Rightarrow$

$$s_n(a_2, a_3, \dots, a_d; a_1) + s_n(a_1, a_3, a_4, \dots, a_d; a_2) + \dots + s_n(a_1, a_2, \dots, a_{d-1}; a_d) = -\text{poly}_{\{a_1, a_2, \dots, a_d\}}(-n)$$

Proof:

- Let (c.f. Exercise 1.31)

$$p_A^\circ(n) = \# \left\{ (m_1, \dots, m_d) \in \mathbb{Z}^d : \begin{array}{l} \text{all } m_j > 0, \\ m_1 a_1 + \dots + m_d a_d = n \end{array} \right\}$$

- $p_A^\circ(n) = (-1)^{d-1} p_A(-n)$ (Ehrhart–McDonald reciprocity)

Rademacher reciprocity (cont'd)

- Then

$$\begin{aligned} (-1)^{d-1} p_A^\circ(n) &= \text{poly}_A(-n) + s_n(a_2, a_3, \dots, a_d; a_1) \\ &\quad + s_n(a_1, a_3, a_4, \dots, a_d; a_2) \\ &\quad + \dots + s_n(a_1, a_2, \dots, a_{d-1}; a_d) \end{aligned}$$

- By definition

$$p_A^\circ(n) = 0 \quad \text{for } n = 1, 2, \dots, a_1 + \dots + a_d - 1$$

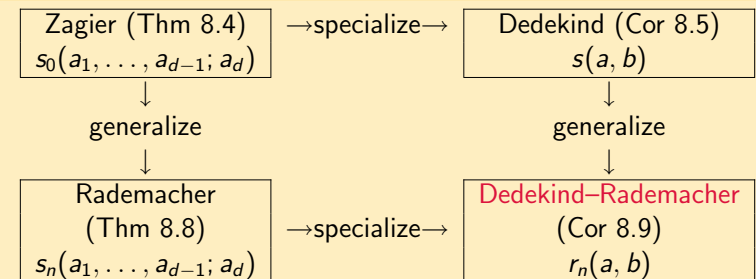
- For those n ,

$$\begin{aligned} 0 &= \text{poly}_A(-n) + s_n(a_2, a_3, \dots, a_d; a_1) \\ &\quad + s_n(a_1, a_3, a_4, \dots, a_d; a_2) \\ &\quad + \dots + s_n(a_1, a_2, \dots, a_{d-1}; a_d) \end{aligned}$$

□

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Relations among reciprocity theorems



Dedekind–Rademacher sums

Definition (Dedekind–Rademacher sum)

$$r_n(a, b) := \sum_{k=0}^{b-1} \left(\left(\frac{ka+n}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right)$$

Note: $r_0(a, b) = s(a, b)$

We're going to see

- reciprocity for Dedekind–Rademacher sums
- where we met Dedekind–Rademacher sums before

Reciprocity law for Dedekind–Rademacher sums

Corollary 8.9 (Reciprocity law for Dedekind–Rademacher sums)

a and b coprime positive integers, $n \in \{1, 2, \dots, a+b\} \Rightarrow$

$$\begin{aligned} r_n(a, b) + r_n(b, a) = & \frac{n^2}{2ab} - \frac{n}{2} \left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b} \right) + \frac{1}{12} \left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right) \\ & + \frac{1}{2} \left(\left(\left(\frac{a^{-1}n}{b} \right) \right) + \left(\left(\frac{b^{-1}n}{a} \right) \right) + \left(\left(\frac{n}{a} \right) \right) + \left(\left(\frac{n}{b} \right) \right) \right) \\ & + \frac{1}{4} (1 + \chi_a(n) + \chi_b(n)), \end{aligned}$$

where $a^{-1}a \equiv 1 \pmod{b}$ and $b^{-1}b \equiv 1 \pmod{a}$

Notation

$$\chi_a(n) := \begin{cases} 1 & \text{if } a|n, \\ 0 & \text{otherwise} \end{cases}$$

Proof of Cor 8.9

Cor 8.9 immediately follows after we see the following lemma

Lemma 8.10

a and b coprime positive integers, $n \in \mathbb{Z} \Rightarrow$

$$r_n(a, b) = -s_n(a, 1; b) + \frac{1}{2} \left(\left(\frac{n}{b} \right) \right) + \frac{1}{2} \left(\left(\frac{na^{-1}}{b} \right) \right) - \frac{1}{4b} + \frac{1}{4} \chi_b(n)$$

Proof of Lem 8.10:

- Recalling the def of a sawtooth function

$$\begin{aligned} \left(\left(\frac{n}{b} \right) \right) &= \begin{cases} 0 & \text{if } b|n, \\ \left\{ \frac{n}{b} \right\} - \frac{1}{2} & \text{otherwise} \end{cases} \\ &= \left\{ \frac{n}{b} \right\} - \frac{1}{2} (1 - \chi_b(n)) \end{aligned}$$

Proof of Lem 8.10 (cont'd)

- Recall: $s_n(a, 1; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{(1 - \xi_b^k)(1 - \xi_b^{ka})}$
- Let $f(n) := \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{1 - \xi_b^k}$ and $g(n) := \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{1 - \xi_b^{ka}}$, then

$$s_n(a, 1; b) = \sum_{m=0}^{b-1} f(n-m)g(m)$$

by the convolution theorem

Proof of Lem 8.10 (further cont'd)

- Then,

$$\begin{aligned}
 f(n) &= -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} && \text{(from Chap 1)} \\
 &= -\left(\left(\frac{-n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b} = \left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b}, \\
 g(n) &= \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{1 - \xi_b^{ka}} = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{ka^{-1}n}}{1 - \xi_b^k} \\
 &= \left(\left(\frac{a^{-1}n}{b}\right)\right) + \frac{1}{2}\chi_b(a^{-1}n) - \frac{1}{2b} \\
 &= \left(\left(\frac{a^{-1}n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b}
 \end{aligned}$$

Proof of Lem 8.10 (further cont'd)

$$\begin{aligned}
 s_n(a, 1; b) &= \sum_{m=0}^{b-1} f(n-m)g(m) \\
 &= \sum_{m=0}^{b-1} \left(\left(\left(\frac{n-m}{b}\right)\right) + \frac{1}{2}\chi_b(n-m) - \frac{1}{2b} \right) \\
 &\quad \times \left(\left(\left(\frac{a^{-1}m}{b}\right)\right) + \frac{1}{2}\chi_b(m) - \frac{1}{2b} \right) \\
 &= - \underbrace{\sum_{m=0}^{b-1} \left(\left(\frac{am+n}{b}\right)\right) \left(\left(\frac{m}{b}\right)\right)}_{r_n(a,b)} + \frac{1}{2} \left(\left(\frac{n}{b}\right)\right) - \frac{1}{4b} + \frac{1}{4}\chi_b(n) \quad \text{(Exer 8.9)}
 \end{aligned}$$

□

Where did we meet Dedekind–Rademacher sums?

Theorem 2.10

For the triangle $\mathcal{T}$ given by (16), where e and f are coprime,

$$\begin{aligned}
 L_{\mathcal{T}}(t) &= \frac{1}{2ef} (tr - u - v)^2 + \frac{1}{2} (tr - u - v) \left(\frac{1}{e} + \frac{1}{f} + \frac{1}{ef} \right) \\
 &\quad + \frac{1}{4} \left(1 + \frac{1}{e} + \frac{1}{f} \right) + \frac{1}{12} \left(\frac{e}{f} + \frac{f}{e} + \frac{1}{ef} \right) \\
 &\quad + \frac{1}{e} \sum_{j=1}^{e-1} \frac{\xi_e^{j(v-tr)}}{(1 - \xi_e^{jf}) (1 - \xi_e^j)} + \frac{1}{f} \sum_{l=1}^{f-1} \frac{\xi_f^{l(u-tr)}}{(1 - \xi_f^{le}) (1 - \xi_f^l)}
 \end{aligned}$$

where $u = \left\lceil \frac{ta}{d} \right\rceil e$ and $v = \left\lceil \frac{tb}{d} \right\rceil f$

Where did we meet Dedekind–Rademacher sums?: Rephrasing

Theorem 2.10, rephrased

For the triangle $\mathcal{T}$ given by (16), where e and f are coprime,

$$\begin{aligned}
 L_{\mathcal{T}}(t) &= \frac{1}{2ef} (tr - u - v)^2 + \frac{1}{2} (tr - u - v) \left(\frac{1}{e} + \frac{1}{f} + \frac{1}{ef} \right) \\
 &\quad + \frac{1}{4} \left(1 + \frac{1}{e} + \frac{1}{f} \right) + \frac{1}{12} \left(\frac{e}{f} + \frac{f}{e} + \frac{1}{ef} \right) \\
 &\quad + s_{v-tr}(f, 1; e) + s_{u-tr}(e, 1; f)
 \end{aligned}$$

where $u = \left\lceil \frac{ta}{d} \right\rceil e$ and $v = \left\lceil \frac{tb}{d} \right\rceil f$

Consequence of reciprocity

The Ehrhart quasipolynomial of a rational convex polygon can be computed efficiently

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Mordell–Pommersheim tetrahedra

Mordell–Pommersheim tetrahedra

- show another type of relations of Dedekind sums with generating functions
- historically first gave rise to the connection of Dedekind sums and lattice-point enumeration in polytopes

- a, b, c positive integers
- **Mordell–Pommersheim tetrahedron**: 3-polytope w/ vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$

$$\mathcal{P} = \left\{ (x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1 \right\}$$

- Let's compute the Ehrhart polynomial of $\mathcal{P}$

The lattice-point enumerator of the MP tetrahedron

- Skip the whole calculation (see Exer 8.11)
- Restrict to pairwise coprime a, b, c (see Exer 8.12)

Theorem 8.11

For the Mordell–Pommersheim tetrahedron $\mathcal{P}$ with a, b, c pairwise coprime,

$$L_{\mathcal{P}}(t) = \frac{abc}{6} t^3 + \frac{ab + ac + bc + 1}{4} t^2 + \left(\frac{3}{4} + \frac{a+b+c}{4} + \frac{1}{12} \left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc} \right) - s(bc, a) - s(ca, b) - s(ab, c) \right) t + 1 \quad \square$$

Thus, the Ehrhart polynomial of the Mordell–Pommersheim tetrahedra can be computed efficiently

Summary

Conclusion

We saw the following can be computed efficiently (by means of reciprocity)

- Dedekind sums
 - Ehrhart polynomials of the Mordell–Pommersheim tetrahedra
- Dedekind–Rademacher sums
 - Ehrhart quasipolynomials of rational convex polygons

Question

Can we compute the Ehrhart quasipolynomial of any convex polytope efficiently?

We try to answer the question in the next lecture