Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(9) Dedekind Sums

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1. Fourier–Dedekind Sums and the Coin-Exchange Problem Revisited
2. The Dedekind Sum and Its Reciprocity and Computational Complexity
3. Rademacher Reciprocity for the Fourier–Dedekind Sum
4. The Mordell–Pommersheim Tetrahedron

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the denominator of $\mathcal{P}$

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

The goal of this chapter

1. Look at Fourier–Dedekind sums more closely
2. In particular, their reciprocity
3. Application to their computation

Reminder: Fourier–Dedekind sums

Definition (Fourier–Dedekind sum)

$$s_n\left(a_1, a_2, \ldots, a_d; b\right) = \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{\left(1-\xi_b^{ka_1}\right)\left(1-\xi_b^{ka_2}\right)\cdots\left(1-\xi_b^{ka_d}\right)}$$

- Note: $\xi_b = e^{2\pi i/b}$

Reminder: Fourier–Dedekind sums

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$$s_n\left(a_1, a_2, \ldots, a_d; b\right) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{\left(1-\xi_b^{ka_1}\right)\left(1-\xi_b^{ka_2}\right)\cdots\left(1-\xi_b^{ka_d}\right)}$$

- Note: $\xi_b = e^{2\pi i/b}$
- Appeared in computation of $p_A(n)$ in Chap 1

Definition (Restricted partition function)

$$p_A(n) = \#\left\{(m_1, \ldots, m_d) \in \mathbb{Z}^d : \begin{array}{l} \text{all } m_j \geq 0, \\ m_1 a_1 + \cdots + m_d a_d = n \end{array}\right\},$$

for $A = \{a_1, \ldots, a_d\}$ a set of coprime positive integers

Restricted partition functions (from Chap 1)

For two or three coins

$$
\begin{aligned}
p_{\{a_1,a_2\}}(n) &= \frac{1}{2a_1} + \frac{1}{2a_2} + \frac{n}{a_1 a_2} + s_{-n}(a_1; a_2) + s_{-n}(a_2; a_1), \\
p_{\{a_1,a_2,a_3\}}(n) &= \frac{n^2}{2a_1 a_2 a_3} + \frac{n}{2}\left(\frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3}\right) \\
&\quad + \frac{1}{12}\left(\frac{3}{a_1} + \frac{3}{a_2} + \frac{3}{a_3} + \frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2}\right) \\
&\quad + s_{-n}(a_1, a_2; a_3) + s_{-n}(a_2, a_3; a_1) + s_{-n}(a_1, a_3; a_2)
\end{aligned}
$$

Example 8.1: When $n = 0$ and $d = 2$

a, b coprime positive integers

$$s_0(a, 1; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{\left(1 - \xi_b^{ka}\right)\left(1 - \xi_b^{k}\right)}$$

Example 8.1: When $n = 0$ and $d = 2$

a, b coprime positive integers

$$s_0(a, 1; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{\left(1 - \xi_b^{ka}\right)\left(1 - \xi_b^{k}\right)}$$

$$= \frac{1}{b} \sum_{k=1}^{b-1} \left(\frac{1}{1 - \xi_b^{ka}} - \frac{1}{2}\right)\left(\frac{1}{1 - \xi_b^{k}} - \frac{1}{2}\right)$$

$$+ \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{k}} + \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{ka}} - \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{4}$$

Example 8.1: When $n = 0$ and $d = 2$

a, b coprime positive integers

$$
\begin{aligned}
s_0(a, 1; b) &= \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{\left(1 - \xi_b^{ka}\right)\left(1 - \xi_b^{k}\right)} \\
&= \frac{1}{b} \sum_{k=1}^{b-1} \left(\frac{1}{1 - \xi_b^{ka}} - \frac{1}{2} \right) \left(\frac{1}{1 - \xi_b^{k}} - \frac{1}{2} \right) \\
&\quad + \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{k}} + \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{ka}} - \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{4} \\
&= \underbrace{\frac{1}{4b} \sum_{k=1}^{b-1} \left(\frac{1 + \xi_b^{ka}}{1 - \xi_b^{ka}} \right) \left(\frac{1 + \xi_b^{k}}{1 - \xi_b^{k}} \right)} + \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{k}} - \frac{b-1}{4b}
\end{aligned}
$$

Example 8.1: When $n = 0$ and $d = 2$

a, b coprime positive integers

$$
\begin{aligned}
s_0(a, 1; b) &= \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{\left(1 - \xi_b^{ka}\right)\left(1 - \xi_b^{k}\right)} \\
&= \frac{1}{b} \sum_{k=1}^{b-1} \left(\frac{1}{1 - \xi_b^{ka}} - \frac{1}{2} \right) \left(\frac{1}{1 - \xi_b^{k}} - \frac{1}{2} \right) \\
&\quad + \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{k}} + \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{ka}} - \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{4} \\
&= \underbrace{\frac{1}{4b} \sum_{k=1}^{b-1} \left(\frac{1 + \xi_b^{ka}}{1 - \xi_b^{ka}} \right) \left(\frac{1 + \xi_b^{k}}{1 - \xi_b^{k}} \right)}_{-s(a,b)} + \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^{k}} - \frac{b-1}{4b}
\end{aligned}
$$

Example 8.1: When $n = 0$ and $d = 2$ (cont'd)

$$s_0(a, 1; b) = -s(a, b) + \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^k} - \frac{b-1}{4b}$$

□

Example 8.1: When $n = 0$ and $d = 2$ (cont'd)

$$s_0(a, 1; b) = -s(a, b) + \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^k} - \frac{b-1}{4b}$$

□

Reminder (from Chap 1)

$$\frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{\left(1 - \xi_b^k\right) \xi_b^{kn}} = -\left\{\frac{n}{b}\right\} + \frac{1}{2} - \frac{1}{2b}$$

Example 8.1: When $n = 0$ and $d = 2$ (cont'd)

$$\begin{aligned} s_0(a, 1; b) &= -s(a, b) + \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{1 - \xi_b^k} - \frac{b-1}{4b} \\ &= -s(a, b) + \frac{1}{2} - \frac{1}{2b} - \frac{b-1}{4b} \end{aligned}$$

□

Reminder (from Chap 1)

$$\frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{\left(1 - \xi_b^k\right) \xi_b^{kn}} = -\left\{\frac{n}{b}\right\} + \frac{1}{2} - \frac{1}{2b}$$

Example 8.1: When $n = 0$ and $d = 2$ (cont'd)

$$\begin{aligned} s_0(a, 1; b) &= -s(a, b) + \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{1-\xi_b^k} - \frac{b-1}{4b} \\ &= -s(a, b) + \frac{1}{2} - \frac{1}{2b} - \frac{b-1}{4b} \\ &= -s(a, b) + \frac{b-1}{4b} \end{aligned}$$

□

Reminder (from Chap 1)

$$\frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{\left(1-\xi_b^k\right)\xi_b^{kn}} = -\left\{\frac{n}{b}\right\} + \frac{1}{2} - \frac{1}{2b}$$

Example 8.2: When $n = 0$ and $d = 2$

Exercise 8.5

For a_1, a_2 coprime to b,

$$s_0\left(a_1, a_2; b\right) = -s\left(a_1 a_2^{-1}, b\right) + \frac{b-1}{4b},$$

where $a_2^{-1} a_2 \equiv 1 \mod b$

How the Fourier–Dedekind sums arose in Chap 1

$a_1, \ldots, a_d$ pairwise coprime

$$f(z) = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_d})\, z^n}$$

How the Fourier–Dedekind sums arose in Chap 1

$a_1, \ldots, a_d$ pairwise coprime

$$f(z) = \frac{1}{(1-z^{a_1})\cdots(1-z^{a_d})\,z^n}$$
$$= \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} + \frac{B_1}{z-1} + \frac{B_2}{(z-1)^2} + \cdots + \frac{B_d}{(z-1)^d}$$
$$+ \sum_{k=1}^{a_1-1} \frac{C_{1k}}{z-\xi_{a_1}^k} + \sum_{k=1}^{a_2-1} \frac{C_{2k}}{z-\xi_{a_2}^k} + \cdots + \sum_{k=1}^{a_d-1} \frac{C_{dk}}{z-\xi_{a_d}^k}$$

How the Fourier–Dedekind sums arose in Chap 1

$a_1, \ldots, a_d$ pairwise coprime

$$
\begin{aligned}
f(z) &= \frac{1}{(1-z^{a_1})\cdots(1-z^{a_d})\, z^n} \\
&= \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} + \frac{B_1}{z-1} + \frac{B_2}{(z-1)^2} + \cdots + \frac{B_d}{(z-1)^d} \\
&\quad + \sum_{k=1}^{a_1-1} \frac{C_{1k}}{z-\xi_{a_1}^k} + \sum_{k=1}^{a_2-1} \frac{C_{2k}}{z-\xi_{a_2}^k} + \cdots + \sum_{k=1}^{a_d-1} \frac{C_{dk}}{z-\xi_{a_d}^k}
\end{aligned}
$$

$p_A(n) = \text{const}\, f(z)$

How the Fourier–Dedekind sums arose in Chap 1

$a_1, \ldots, a_d$ pairwise coprime

$$
\begin{aligned}
f(z) &= \frac{1}{(1-z^{a_1})\cdots(1-z^{a_d})\, z^n} \\
&= \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} + \frac{B_1}{z-1} + \frac{B_2}{(z-1)^2} + \cdots + \frac{B_d}{(z-1)^d} \\
&\quad + \sum_{k=1}^{a_1-1} \frac{C_{1k}}{z-\xi_{a_1}^k} + \sum_{k=1}^{a_2-1} \frac{C_{2k}}{z-\xi_{a_2}^k} + \cdots + \sum_{k=1}^{a_d-1} \frac{C_{dk}}{z-\xi_{a_d}^k}
\end{aligned}
$$

$$
\begin{aligned}
p_A(n) &= \mathrm{const}\, f(z) \\
&= -B_1 + B_2 - \cdots + (-1)^d B_d + s_{-n}(a_2, a_3, \ldots, a_d; a_1) \\
&\quad + s_{-n}(a_1, a_3, a_4, \ldots, a_d; a_2) + \cdots + s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d)
\end{aligned}
$$

How the Fourier–Dedekind sums arose in Chap 1

$a_1, \ldots, a_d$ pairwise coprime

$$
\begin{aligned}
f(z) &= \frac{1}{(1-z^{a_1})\cdots(1-z^{a_d})\, z^n} \\
&= \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} + \frac{B_1}{z-1} + \frac{B_2}{(z-1)^2} + \cdots + \frac{B_d}{(z-1)^d} \\
&\quad + \sum_{k=1}^{a_1-1} \frac{C_{1k}}{z-\xi_{a_1}^k} + \sum_{k=1}^{a_2-1} \frac{C_{2k}}{z-\xi_{a_2}^k} + \cdots + \sum_{k=1}^{a_d-1} \frac{C_{dk}}{z-\xi_{a_d}^k}
\end{aligned}
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&\quad + s_{-n}(a_1, a_3, a_4, \ldots, a_d; a_2) + \cdots + s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d)
\end{aligned}
$$

Note: $B_1, \ldots, B_d$ are polynomials in n (Exer, 8.6)

The polynomial part of $p_A(n)$

Definition (Polynomial part)

The polynomial part of $p_A(n)$ is

$$\mathrm{poly}_A(n) := -B_1 + B_2 - \cdots + (-1)^d B_d$$

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Example: $n = 2$

$$p_{\{a_1,a_2\}}(n) = \frac{1}{2a_1} + \frac{1}{2a_2} + \frac{n}{a_1 a_2} + s_{-n}(a_1; a_2) + s_{-n}(a_2; a_1)$$

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Example: $n = 2$

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Therefore

$$\mathrm{poly}_{\{a_1,a_2\}}(n) = \frac{1}{2a_1} + \frac{1}{2a_2} + \frac{n}{a_1 a_2}$$

Zagier's reciprocity

Theorem 8.4 (Zagier's reciprocity)

For any pairwise coprime positive integers $a_1, a_2, \ldots, a_d$,

$$s_0(a_2, a_3, \ldots, a_d; a_1) + s_0(a_1, a_3, a_4, \ldots, a_d; a_2) + \cdots + s_0(a_1, a_2, \ldots, a_{d-1}; a_d) = 1 - \mathrm{poly}_{\{a_1, a_2, \ldots, a_d\}}(0)$$

Example: $d = 2$

$$s_0(a_1; a_2) + s_0(a_2; a_1) = 1 - \frac{1}{2a_1} - \frac{1}{2a_2}$$

Proof of Zagier's reciprocity

- By computing the constant term of $p_A(n)$

$$
\begin{aligned}
p_A(0) = \mathrm{poly}_A(0) \\
&+ s_0\left(a_2, a_3, \dots, a_d; a_1\right) \\
&+ s_0\left(a_1, a_3, a_4, \dots, a_d; a_2\right) \\
&+ \cdots + s_0\left(a_1, a_2, \dots, a_{d-1}; a_d\right)
\end{aligned}
$$

Proof of Zagier's reciprocity

- By computing the constant term of $p_A(n)$

$$\begin{aligned} p_A(0) = \operatorname{poly}_A(0) & \\ &+ s_0\left(a_2, a_3, \ldots, a_d; a_1\right) \\ &+ s_0\left(a_1, a_3, a_4, \ldots, a_d; a_2\right) \\ &+ \cdots + s_0\left(a_1, a_2, \ldots, a_{d-1}; a_d\right) \end{aligned}$$

- $p_A(0) = 1$ (Exer 3.27)

Proof of Zagier's reciprocity

- By computing the constant term of $p_A(n)$

$$\begin{aligned} p_A(0) = \operatorname{poly}_A(0) & \\ &+ s_0\left(a_2, a_3, \dots, a_d; a_1\right) \\ &+ s_0\left(a_1, a_3, a_4, \dots, a_d; a_2\right) \\ &+ \cdots + s_0\left(a_1, a_2, \dots, a_{d-1}; a_d\right) \end{aligned}$$

- $p_A(0) = 1$ (Exer 3.27)
- $\therefore$

$$\begin{aligned} 1 = \operatorname{poly}_A(0) & \\ &+ s_0\left(a_2, a_3, \dots, a_d; a_1\right) \\ &+ s_0\left(a_1, a_3, a_4, \dots, a_d; a_2\right) \\ &+ \cdots + s_0\left(a_1, a_2, \dots, a_{d-1}; a_d\right) \end{aligned}$$

□

Contents of this section

Relations among reciprocity theorems

Zagier (Thm 8.4) $s_0(a_1, \dots, a_{d-1}; a_d)$	$\longrightarrow$specialize$\longrightarrow$	Dedekind (Cor 8.5) $s(a, b)$
$\downarrow$ generalize $\downarrow$		$\downarrow$ generalize $\downarrow$
Rademacher (Thm 8.8) $s_n(a_1, \dots, a_{d-1}; a_d)$	$\longrightarrow$specialize$\longrightarrow$	Dedekind–Rademacher (Cor 8.9) $r_n(a, b)$

Reciprocity of the Dedekind sums

Corollary 8.5 (Dedekind's reciprocity law)

For any coprime positive integers a and b,

$$s(a, b) + s(b, a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}$$

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Proof:

- $s_0(a,1;b) = -s(a,b) + \frac{b-1}{4b}$ (Example 8.1)

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Proof:

- $s_0(a,1;b) = -s(a,b) + \frac{b-1}{4b}$ (Example 8.1)
- $s_0(b,a;1) = 0$ (by Def)

Reciprocity of the Dedekind sums

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Proof:

- $s_0(a,1;b) = -s(a,b) + \frac{b-1}{4b}$ (Example 8.1)
- $s_0(b,a;1) = 0$ (by Def)
- With Zagier's reciprocity

$$\begin{aligned} &s_0(a,1;b) + s_0(b,a;1) + s_0(1,b;a) \\ &\quad = 1 - \frac{1}{12}\left(\frac{3}{a} + 3 + \frac{3}{b} + \frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) \end{aligned}$$

□

Computational aspect

A consequence of Dedekind's recprocity law

It enables us to compute Dedekind sums efficiently

- Reminder: $s(a, b) = s(a \bmod b, b)$

Example 8.6: $a = 100, b = 147$

$s(100, 147)$

Computational aspect

A consequence of Dedekind's recprocity law

It enables us to compute Dedekind sums efficiently

- Reminder: $s(a,b) = s(a \bmod b, b)$

Example 8.6: $a = 100, b = 147$

$$s(100,147) = \frac{1}{12}\left(\frac{100}{147} + \frac{147}{100} + \frac{1}{14700}\right) - \frac{1}{4} - s(147,100)$$

Computational aspect

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- Reminder: $s(a, b) = s(a \bmod b, b)$

Example 8.6: $a = 100, b = 147$

$$s(100, 147) = \frac{1}{12}\left(\frac{100}{147} + \frac{147}{100} + \frac{1}{14700}\right) - \frac{1}{4} - s(147, 100)$$
$$= -\frac{1249}{17640} - s(47, 100)$$

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It enables us to compute Dedekind sums efficiently

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Example 8.6: $a = 100, b = 147$

$$s(100, 147) = \frac{1}{12}\left(\frac{100}{147} + \frac{147}{100} + \frac{1}{14700}\right) - \frac{1}{4} - s(147, 100)$$
$$= -\frac{1249}{17640} - s(47, 100)$$
$$= -\frac{1249}{17640} - \left(\frac{1}{12}\left(\frac{47}{100} + \frac{100}{47} + \frac{1}{4700}\right) - \frac{1}{4} - s(100, 47)\right)$$

Computational aspect

A consequence of Dedekind's recprocity law

It enables us to compute Dedekind sums efficiently

- Reminder: $s(a, b) = s(a \bmod b, b)$

Example 8.6: $a = 100, b = 147$

$$\begin{aligned} s(100, 147) &= \frac{1}{12}\left(\frac{100}{147} + \frac{147}{100} + \frac{1}{14700}\right) - \frac{1}{4} - s(147, 100) \\ &= -\frac{1249}{17640} - s(47, 100) \\ &= -\frac{1249}{17640} - \left(\frac{1}{12}\left(\frac{47}{100} + \frac{100}{47} + \frac{1}{4700}\right) - \frac{1}{4} - s(100, 47)\right) \\ &= -\frac{773}{20727} + s(6, 47) \end{aligned}$$

Computational aspect (cont'd)

$$s(100, 147) = -\frac{773}{20727} + \frac{1}{12}\left(\frac{6}{47} + \frac{47}{6} + \frac{1}{282}\right) - \frac{1}{4} - s(47, 6)$$

Computational aspect (cont'd)

$$
\begin{aligned}
s(100,147) &= -\frac{773}{20727} + \frac{1}{12}\left(\frac{6}{47} + \frac{47}{6} + \frac{1}{282}\right) - \frac{1}{4} - s(47,6) \\
&= \frac{166}{441} - s(5,6)
\end{aligned}
$$

Computational aspect (cont'd)

$$s(100,147) = -\frac{773}{20727} + \frac{1}{12}\left(\frac{6}{47} + \frac{47}{6} + \frac{1}{282}\right) - \frac{1}{4} - s(47,6)$$
$$= \frac{166}{441} - s(5,6)$$
$$= \frac{166}{441} - \left(\frac{1}{12}\left(\frac{5}{6} + \frac{6}{5} + \frac{1}{30}\right) - \frac{1}{4} - s(6,5)\right)$$

Computational aspect (cont'd)

$$
\begin{aligned}
s(100,147) &= -\frac{773}{20727} + \frac{1}{12}\left(\frac{6}{47} + \frac{47}{6} + \frac{1}{282}\right) - \frac{1}{4} - s(47,6) \\
&= \frac{166}{441} - s(5,6) \\
&= \frac{166}{441} - \left(\frac{1}{12}\left(\frac{5}{6} + \frac{6}{5} + \frac{1}{30}\right) - \frac{1}{4} - s(6,5)\right) \\
&= \frac{2003}{4410} + s(1,5)
\end{aligned}
$$

Computational aspect (cont'd)

$$\begin{aligned}
s(100,147) &= -\frac{773}{20727} + \frac{1}{12}\left(\frac{6}{47} + \frac{47}{6} + \frac{1}{282}\right) - \frac{1}{4} - s(47,6) \\
&= \frac{166}{441} - s(5,6) \\
&= \frac{166}{441} - \left(\frac{1}{12}\left(\frac{5}{6} + \frac{6}{5} + \frac{1}{30}\right) - \frac{1}{4} - s(6,5)\right) \\
&= \frac{2003}{4410} + s(1,5) \\
&= \frac{2003}{4410} - \frac{1}{4} + \frac{1}{30} + \frac{5}{12}
\end{aligned}$$

where we used $s(1,k) = -\frac{1}{4} + \frac{1}{6k} + \frac{k}{12}$ (Exer 7.20)

Computational aspect (cont'd)

$$
\begin{aligned}
s(100,147) &= -\frac{773}{20727} + \frac{1}{12}\left(\frac{6}{47} + \frac{47}{6} + \frac{1}{282}\right) - \frac{1}{4} - s(47,6) \\
&= \frac{166}{441} - s(5,6) \\
&= \frac{166}{441} - \left(\frac{1}{12}\left(\frac{5}{6} + \frac{6}{5} + \frac{1}{30}\right) - \frac{1}{4} - s(6,5)\right) \\
&= \frac{2003}{4410} + s(1,5) \\
&= \frac{2003}{4410} - \frac{1}{4} + \frac{1}{30} + \frac{5}{12} \\
&= \frac{577}{882}
\end{aligned}
$$

where we used $s(1,k) = -\frac{1}{4} + \frac{1}{6k} + \frac{k}{12}$ (Exer 7.20)

Contents of this section

Relations among reciprocity theorems

Zagier (Thm 8.4) $s_0(a_1, \dots, a_{d-1}; a_d)$	→specialize→	Dedekind (Cor 8.5) $s(a, b)$
↓ generalize ↓		↓ generalize ↓
Rademacher (Thm 8.8) $s_n(a_1, \dots, a_{d-1}; a_d)$	→specialize→	Dedekind–Rademacher (Cor 8.9) $r_n(a, b)$

Rademacher reciprocity

Theorem 8.8 (Rademacher reciprocity)

$a_1, a_2, \ldots, a_d$ pairwise coprime positive integers
$n \in \{1, 2, \ldots, a_1 + \cdots + a_d - 1\} \Rightarrow$

$$s_n(a_2, a_3, \ldots, a_d; a_1) + s_n(a_1, a_3, a_4, \ldots, a_d; a_2) + \cdots + s_n(a_1, a_2, \ldots, a_{d-1}; a_d) = -\operatorname{poly}_{\{a_1, a_2, \ldots, a_d\}}(-n)$$

Rademacher reciprocity

Theorem 8.8 (Rademacher reciprocity)

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$n \in \{1, 2, \ldots, a_1 + \cdots + a_d - 1\} \Rightarrow$

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Proof:

- Let (c.f. Exercise 1.31)

$$p_A^{\circ}(n) = \#\left\{(m_1, \ldots, m_d) \in \mathbb{Z}^d : \begin{array}{l} \text{all } m_j > 0, \\ m_1 a_1 + \cdots + m_d a_d = n \end{array}\right\}$$

Rademacher reciprocity

Theorem 8.8 (Rademacher reciprocity)

$a_1, a_2, \ldots, a_d$ pairwise coprime positive integers
$n \in \{1, 2, \ldots, a_1 + \cdots + a_d - 1\} \Rightarrow$

$$s_n(a_2, a_3, \ldots, a_d; a_1) + s_n(a_1, a_3, a_4, \ldots, a_d; a_2) + \cdots + s_n(a_1, a_2, \ldots, a_{d-1}; a_d) = -\operatorname{poly}_{\{a_1, a_2, \ldots, a_d\}}(-n)$$

Proof:

- Let (c.f. Exercise 1.31)

$$p_A^\circ(n) = \#\left\{(m_1, \ldots, m_d) \in \mathbb{Z}^d : \begin{array}{l} \text{all } m_j > 0, \\ m_1 a_1 + \cdots + m_d a_d = n \end{array}\right\}$$

- $p_A^\circ(n) = (-1)^{d-1} p_A(-n)$ (Ehrhart–Mcdonald reciprocity)

Rademacher reciprocity (cont'd)

- Then

$$(-1)^{d-1} p_A^\circ(n) = \mathrm{poly}_A(-n) + s_n\left(a_2, a_3, \ldots, a_d; a_1\right) + s_n\left(a_1, a_3, a_4, \ldots, a_d; a_2\right) + \cdots + s_n\left(a_1, a_2, \ldots, a_{d-1}; a_d\right)$$

Rademacher reciprocity (cont'd)

- Then

$$\begin{aligned}(-1)^{d-1} p_A^{\circ}(n) = {} & \mathrm{poly}_A(-n) + s_n(a_2, a_3, \ldots, a_d; a_1) \\ & + s_n(a_1, a_3, a_4, \ldots, a_d; a_2) \\ & + \cdots + s_n(a_1, a_2, \ldots, a_{d-1}; a_d)\end{aligned}$$

- By definition

$$p_A^{\circ}(n) = 0 \qquad \text{for} \quad n = 1, 2, \ldots, a_1 + \cdots + a_d - 1$$

Rademacher reciprocity (cont'd)

- Then

$$\begin{aligned}(-1)^{d-1}p_A^\circ(n) = \mathrm{poly}_A(-n) &+ s_n\left(a_2, a_3, \ldots, a_d; a_1\right) \\ &+ s_n\left(a_1, a_3, a_4, \ldots, a_d; a_2\right) \\ &+ \cdots + s_n\left(a_1, a_2, \ldots, a_{d-1}; a_d\right)\end{aligned}$$

- By definition

$$p_A^\circ(n) = 0 \qquad \text{for} \quad n = 1, 2, \ldots, a_1 + \cdots + a_d - 1$$

- For those n,

$$\begin{aligned}0 = \mathrm{poly}_A(-n) &+ s_n\left(a_2, a_3, \ldots, a_d; a_1\right) \\ &+ s_n\left(a_1, a_3, a_4, \ldots, a_d; a_2\right) \\ &+ \cdots + s_n\left(a_1, a_2, \ldots, a_{d-1}; a_d\right)\end{aligned}$$

□

Contents of this section

Relations among reciprocity theorems

Zagier (Thm 8.4) $s_0(a_1, \ldots, a_{d-1}; a_d)$	→specialize→	Dedekind (Cor 8.5) $s(a, b)$
↓ generalize ↓		↓ generalize ↓
Rademacher (Thm 8.8) $s_n(a_1, \ldots, a_{d-1}; a_d)$	→specialize→	Dedekind–Rademacher (Cor 8.9) $r_n(a, b)$

Dedekind–Rademacher sums

Definition (Dedekind–Rademacher sum)

$$r_n(a,b) := \sum_{k=0}^{b-1} \left(\left(\frac{ka+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

Note: $r_0(a,b) = s(a,b)$

We're going to see

- reciprocity for Dedekind–Rademacher sums
- where we met Dedekind–Rademacher sums before

Reciprocity law for Dedekind–Rademacher sums

Corollary 8.9 (Reciprocity law for Dedekind–Rademacher sums)

a and b coprime positive integers, $n \in \{1, 2, \ldots, a+b\} \Rightarrow$

$$
\begin{aligned}
r_n(a,b) + r_n(b,a) = &\frac{n^2}{2ab} - \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b}\right) + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) \\
&+ \frac{1}{2}\left(\left(\left(\frac{a^{-1}n}{b}\right)\right) + \left(\left(\frac{b^{-1}n}{a}\right)\right) + \left(\left(\frac{n}{a}\right)\right) + \left(\left(\frac{n}{b}\right)\right)\right) \\
&+ \frac{1}{4}\left(1 + \chi_a(n) + \chi_b(n)\right),
\end{aligned}
$$

where $a^{-1}a \equiv 1 \mod b$ and $b^{-1}b \equiv 1 \mod a$

Notation

$$
\chi_a(n) := \begin{cases} 1 & \text{if } a|n, \\ 0 & \text{otherwise} \end{cases}
$$

Proof of Cor 8.9

Cor 8.9 immediately follows after we see the following lemma

Lemma 8.10

a and b coprime positive integers, $n \in \mathbb{Z} \Rightarrow$

$$r_n(a,b) = -s_n(a,1;b) + \frac{1}{2}\left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\left(\left(\frac{na^{-1}}{b}\right)\right) - \frac{1}{4b} + \frac{1}{4}\chi_b(n)$$

Proof of Cor 8.9

Cor 8.9 immediately follows after we see the following lemma

Lemma 8.10

a and b coprime positive integers, $n \in \mathbb{Z} \Rightarrow$

$$r_n(a,b) = -s_n(a,1;b) + \frac{1}{2}\left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\left(\left(\frac{na^{-1}}{b}\right)\right) - \frac{1}{4b} + \frac{1}{4}\chi_b(n)$$

Proof of Lem 8.10:

- Recalling the def of a sawtooth function

$$\left(\left(\frac{n}{b}\right)\right) = \begin{cases} 0 & \text{if } b|n, \\ \left\{\frac{n}{b}\right\} - \frac{1}{2} & \text{otherwise} \end{cases}$$

Proof of Cor 8.9

Cor 8.9 immediately follows after we see the following lemma

Lemma 8.10

a and b coprime positive integers, $n \in \mathbb{Z} \Rightarrow$

$$r_n(a,b) = -s_n(a,1;b) + \frac{1}{2}\left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\left(\left(\frac{na^{-1}}{b}\right)\right) - \frac{1}{4b} + \frac{1}{4}\chi_b(n)$$

Proof of Lem 8.10:

- Recalling the def of a sawtooth function

$$\left(\left(\frac{n}{b}\right)\right) = \begin{cases} 0 & \text{if } b|n, \\ \left\{\frac{n}{b}\right\} - \frac{1}{2} & \text{otherwise} \end{cases}$$
$$= \left\{\frac{n}{b}\right\} - \frac{1}{2}(1 - \chi_b(n))$$

Proof of Lem 8.10 (cont'd)

- Recall: $s_n(a, 1; b) = \dfrac{1}{b} \displaystyle\sum_{k=1}^{b-1} \dfrac{\xi_b^{kn}}{\left(1 - \xi_b^{k}\right)\left(1 - \xi_b^{ka}\right)}$
- Let $f(n) := \dfrac{1}{b} \displaystyle\sum_{k=1}^{b-1} \dfrac{\xi_b^{kn}}{1 - \xi_b^{k}}$ and $g(n) := \dfrac{1}{b} \displaystyle\sum_{k=1}^{b-1} \dfrac{\xi_b^{kn}}{1 - \xi_b^{ka}}$, then

$$s_n(a, 1; b) = \sum_{m=0}^{b-1} f(n-m)\, g(m)$$

by the convolution theorem

Proof of Lem 8.10 (further cont'd)

- Then,

$$f(n) = -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} \qquad \text{(from Chap 1)}$$

Proof of Lem 8.10 (further cont'd)

- Then,

$$f(n) = -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} \qquad \text{(from Chap 1)}$$
$$= -\left(\left(\frac{-n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b}$$

Proof of Lem 8.10 (further cont'd)

- Then,

$$f(n) = -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} \qquad \text{(from Chap 1)}$$
$$= -\left(\left(\frac{-n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b} = \left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b},$$

Proof of Lem 8.10 (further cont'd)

- Then,

$$\begin{aligned} f(n) &= -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} \qquad \text{(from Chap 1)} \\ &= -\left(\left(\frac{-n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b} = \left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b}, \end{aligned}$$

$$g(n) = \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{1-\xi_b^{ka}}$$

Proof of Lem 8.10 (further cont'd)

- Then,

$$\begin{aligned} f(n) &= -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} \qquad \text{(from Chap 1)} \\ &= -\left(\left(\frac{-n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b} = \left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b}, \\ g(n) &= \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{1-\xi_b^{ka}} = \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{ka^{-1}n}}{1-\xi_b^{k}} \end{aligned}$$

Proof of Lem 8.10 (further cont'd)

- Then,

$$f(n) = -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} \qquad \text{(from Chap 1)}$$
$$= -\left(\left(\frac{-n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b} = \left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b},$$

$$g(n) = \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{1-\xi_b^{ka}} = \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{ka^{-1}n}}{1-\xi_b^{k}}$$
$$= \left(\left(\frac{a^{-1}n}{b}\right)\right) + \frac{1}{2}\chi_b\left(a^{-1}n\right) - \frac{1}{2b}$$

Proof of Lem 8.10 (further cont'd)

- Then,

$$f(n) = -\left\{\frac{-n}{b}\right\} + \frac{1}{2} - \frac{1}{2b} \qquad \text{(from Chap 1)}$$

$$= -\left(\left(\frac{-n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b} = \left(\left(\frac{n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b},$$

$$g(n) = \frac{1}{b}\sum_{k=1}^{b-1}\frac{\xi_b^{kn}}{1-\xi_b^{ka}} = \frac{1}{b}\sum_{k=1}^{b-1}\frac{\xi_b^{ka^{-1}n}}{1-\xi_b^{k}}$$

$$= \left(\left(\frac{a^{-1}n}{b}\right)\right) + \frac{1}{2}\chi_b\left(a^{-1}n\right) - \frac{1}{2b}$$

$$= \left(\left(\frac{a^{-1}n}{b}\right)\right) + \frac{1}{2}\chi_b(n) - \frac{1}{2b}$$

Proof of Lem 8.10 (further cont'd)

$$s_n(a, 1; b) = \sum_{m=0}^{b-1} f(n-m)\, g(m)$$

Proof of Lem 8.10 (further cont'd)

$$
\begin{aligned}
s_n(a, 1; b) &= \sum_{m=0}^{b-1} f(n-m)\, g(m) \\
&= \sum_{m=0}^{b-1} \left(\left(\left(\frac{n-m}{b} \right) \right) + \frac{1}{2} \chi_b(n-m) - \frac{1}{2b} \right) \\
&\qquad \times \left(\left(\left(\frac{a^{-1} m}{b} \right) \right) + \frac{1}{2} \chi_b(m) - \frac{1}{2b} \right)
\end{aligned}
$$

Proof of Lem 8.10 (further cont'd)

$$
\begin{aligned}
s_n(a,1;b) &= \sum_{m=0}^{b-1} f(n-m)\, g(m) \\
&= \sum_{m=0}^{b-1} \left(\left(\left(\frac{n-m}{b} \right)\right) + \frac{1}{2}\chi_b(n-m) - \frac{1}{2b} \right) \\
&\qquad \times \left(\left(\left(\frac{a^{-1}m}{b} \right)\right) + \frac{1}{2}\chi_b(m) - \frac{1}{2b} \right) \\
&= \underbrace{-\sum_{m=0}^{b-1} \left(\left(\frac{am+n}{b} \right)\right) \left(\left(\frac{m}{b} \right)\right)} + \frac{1}{2}\left(\left(\frac{a^{-1}n}{b} \right)\right) \\
&\qquad + \frac{1}{2}\left(\left(\frac{n}{b} \right)\right) - \frac{1}{4b} + \frac{1}{4}\chi_b(n) \qquad \text{(Exer 8.9)}
\end{aligned}
$$

□

Proof of Lem 8.10 (further cont'd)

$$
\begin{aligned}
s_n(a,1;b) &= \sum_{m=0}^{b-1} f(n-m)\,g(m) \\
&= \sum_{m=0}^{b-1} \left(\left(\left(\frac{n-m}{b} \right)\right) + \frac{1}{2}\chi_b(n-m) - \frac{1}{2b} \right) \\
&\qquad \times \left(\left(\left(\frac{a^{-1}m}{b} \right)\right) + \frac{1}{2}\chi_b(m) - \frac{1}{2b} \right) \\
&= \underbrace{-\sum_{m=0}^{b-1} \left(\left(\frac{am+n}{b} \right)\right) \left(\left(\frac{m}{b} \right)\right)}_{r_n(a,b)} + \frac{1}{2} \left(\left(\frac{a^{-1}n}{b} \right)\right) \\
&\qquad + \frac{1}{2}\left(\left(\frac{n}{b} \right)\right) - \frac{1}{4b} + \frac{1}{4}\chi_b(n) \qquad \text{(Exer 8.9)}
\end{aligned}
$$

Where did we meet Dedekind–Rademacher sums?

Theorem 2.10

For the triangle $\mathcal{T}$ given by (16), where e and f are coprime,

$$
\begin{aligned}
L_{\mathcal{T}}(t) &= \frac{1}{2ef}(tr-u-v)^2 + \frac{1}{2}(tr-u-v)\left(\frac{1}{e}+\frac{1}{f}+\frac{1}{ef}\right) \\
&+ \frac{1}{4}\left(1+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f}+\frac{f}{e}+\frac{1}{ef}\right) \\
&+ \frac{1}{e}\sum_{j=1}^{e-1}\frac{\xi_e^{j(v-tr)}}{\left(1-\xi_e^{jf}\right)\left(1-\xi_e^{j}\right)} + \frac{1}{f}\sum_{l=1}^{f-1}\frac{\xi_f^{l(u-tr)}}{\left(1-\xi_f^{le}\right)\left(1-\xi_f^{l}\right)}
\end{aligned}
$$

where $u = \left\lceil \frac{ta}{d} \right\rceil e$ and $v = \left\lceil \frac{tb}{d} \right\rceil f$

Where did we meet Dedekind–Rademacher sums?: Rephrasing

Theorem 2.10, rephrased

For the triangle $\mathcal{T}$ given by (16), where e and f are coprime,

$$
\begin{aligned}
L_{\mathcal{T}}(t) = {} & \frac{1}{2ef}(tr - u - v)^2 + \frac{1}{2}(tr - u - v)\left(\frac{1}{e} + \frac{1}{f} + \frac{1}{ef}\right) \\
& + \frac{1}{4}\left(1 + \frac{1}{e} + \frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f} + \frac{f}{e} + \frac{1}{ef}\right) \\
& + s_{v-tr}(f, 1; e) + s_{u-tr}(e, 1; f)
\end{aligned}
$$

where $u = \left\lceil \frac{ta}{d} \right\rceil e$ and $v = \left\lceil \frac{tb}{d} \right\rceil f$

Where did we meet Dedekind–Rademacher sums?: Rephrasing

Theorem 2.10, rephrased

For the triangle $\mathcal{T}$ given by (16), where e and f are coprime,

$$L_{\mathcal{T}}(t) = \frac{1}{2ef}(tr-u-v)^2 + \frac{1}{2}(tr-u-v)\left(\frac{1}{e}+\frac{1}{f}+\frac{1}{ef}\right) + \frac{1}{4}\left(1+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f}+\frac{f}{e}+\frac{1}{ef}\right) + s_{v-tr}(f,1;e) + s_{u-tr}(e,1;f)$$

where $u = \left\lceil \frac{ta}{d} \right\rceil e$ and $v = \left\lceil \frac{tb}{d} \right\rceil f$

Consequence of reciprocity

The Ehrhart quasipolynomial of a rational convex polygon can be computed efficiently

Mordell–Pommersheim tetrahedra

Mordell–Pommersheim tetrahedra

- show another type of relations of Dedekind sums with generating functions
- historically first gave rise to the connection of Dedekind sums and lattice-point enumeration in polytopes

- a, b, c positive integers
- Morderll–Pommersheim tetrahedron: 3-polytope w/ vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$

$$\mathcal{P} = \left\{(x,y,z) \in \mathbb{R}^3 : x,y,z \geq 0,\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\right\}$$

- Let's compute the Ehrhart polynomial of $\mathcal{P}$

The lattice-point enumerator of the MP tetrahedron

- Skip the whole calculation (see Exer 8.11)
- Restrict to pairwise coprime a, b, c (see Exer 8.12)

Theorem 8.11

For the Mordell–Pommersheim tetrahedron $\mathcal{P}$ with a, b, c pairwise coprime,

$$L_{\mathcal{P}}(t) = \frac{abc}{6}\,t^3 + \frac{ab+ac+bc+1}{4}\,t^2 + \left(\frac{3}{4} + \frac{a+b+c}{4} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc}\right) - s(bc,a) - s(ca,b) - s(ab,c)\right) t + 1 \qquad \square$$

Thus, the Ehrhart polynomial of the Mordell–Pommersheim tetrahedra can be computed efficiently

Summary

Conclusion

We saw the following can be computed efficiently (by means of reciprocity)

- Dedekind sums
  - Ehrhart polynomials of the Mordell–Pommersheim tetrahedra
- Dedekind–Rademacher sums
  - Ehrhart quasipolynomials of rational convex polygons

Summary

Conclusion

We saw the following can be computed efficiently (by means of reciprocity)

- Dedekind sums
  - Ehrhart polynomials of the Mordell–Pommersheim tetrahedra
- Dedekind–Rademacher sums
  - Ehrhart quasipolynomials of rational convex polygons

Question

Can we compute the Ehrhart quasipolynomial of any convex polytope efficiently?

We try to answer the question in the next lecture