Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(8) Finite Fourier Analysis

Yoshio Okamoto

Tokyo Institute of Technology

June 18, 2009

"Last updated: 2009/06/17 15:25"

1. A Motivating Example
2. Finite Fourier Series for Periodic Functions on $\mathbb{Z}$
3. The Finite Fourier Transform and Its Properties
4. The Parseval Identity
5. The Convolution of Finite Fourier Series

Important theorems from the previous lectures

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the denominator of $\mathcal{P}$

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

The goal of this chapter

Motivation

- Quasipolynomials involve periodic functions
- Especially Fourier–Dedekind sums

The goal of this chapter

1. Establish a theory for periodic functions
2. Prepare for the next chapter

1. A Motivating Example
2. Finite Fourier Series for Periodic Functions on $\mathbb{Z}$
3. The Finite Fourier Transform and Its Properties
4. The Parseval Identity
5. The Convolution of Finite Fourier Series

Example 7.1

- Consider the following function with period 3

$$\begin{aligned} n &: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots \\ a(n) &: 1, 5, 2, 1, 5, 2, 1, 5, 2, 1, \ldots \end{aligned}$$

- Embed this sequence into a generating function as follows

$$F(z) := 1 + 5z + 2z^2 + z^3 + 5z^4 + 2z^5 + \cdots = \sum_{n \geq 0} a(n)\, z^n$$

- Then

$$\begin{aligned} F(z) &= 1+5z+2z^2+z^3\left(1+5z+2z^2\right)+z^6\left(1+5z+2z^2\right)+\cdots \\ &= \left(1+5z+2z^2\right)\sum_{k\geq 0} z^{3k} = \frac{1+5z+2z^2}{1-z^3} \end{aligned}$$

Example 7.1 (cont'd)

- By the partial fraction expansion we obtain

$$F(z) = \frac{\hat{a}(0)}{1-z} + \frac{\hat{a}(1)}{1-\rho z} + \frac{\hat{a}(2)}{1-\rho^2 z}$$

  - the constants $\hat{a}(0), \hat{a}(1), \hat{a}(2)$ remain to be found
  - $\rho := e^{2\pi i/3}$

- Then

$$\begin{aligned} F(z) &= \hat{a}(0)\sum_{n\geq 0} z^n + \hat{a}(1)\sum_{n\geq 0}(\rho z)^n + \hat{a}(2)\sum_{n\geq 0}(\rho^2 z)^n \\ &= \sum_{n\geq 0}\left(\hat{a}(0) + \hat{a}(1)\rho^n + \hat{a}(2)\rho^{2n}\right) z^n \end{aligned}$$

We've derived the finite Fourier series of our sequence $a(n)$!

Example 7.1 (further cont'd)

- To obtain $\hat{a}(0), \hat{a}(1), \hat{a}(2)$, we use the identify for all $z \in \mathbb{C}$

$$\begin{aligned} 1 + 5z + 2z^2 = \hat{a}(0)\,(1-\rho z)\,(1-\rho^2 z) &+ \hat{a}(1)\,(1-z)\,(1-\rho^2 z) \\ &+ \hat{a}(2)\,(1-z)\,(1-\rho z) \end{aligned}$$

- By letting $z = 1,\ \rho^2,\ \rho$, respectively, we obtain

$$\begin{aligned} 1 + 5 + 2 &= 3\,\hat{a}(0) \\ 1 + 5\rho^2 + 2\rho^4 &= 3\,\hat{a}(1) \\ 1 + 5\rho + 2\rho^2 &= 3\,\hat{a}(2) \end{aligned}$$

  where we used the identity $(1-\rho)(1-\rho^2) = 3$ (Exer 7.2)

- Then, we get the finite Fourier series for our sequence

$$a(n) = \frac{8}{3} + \left(-\frac{4}{3} - \rho\right)\rho^n + \left(-\frac{1}{3} + \rho\right)\rho^{2n} \qquad \square$$

1. A Motivating Example
2. Finite Fourier Series for Periodic Functions on $\mathbb{Z}$
3. The Finite Fourier Transform and Its Properties
4. The Parseval Identity
5. The Convolution of Finite Fourier Series

Toward the general theory (1)

- $\{a(n)\}_{n=0}^{\infty}$ any periodic sequence on $\mathbb{Z}$ of period b
- Throughout the chapter, we fix $\xi := e^{2\pi i/b}$
- Embed $\{a(n)\}_{n=0}^{\infty}$ into a generating function

$$F(z) := \sum_{n\geq 0} a(n)\, z^n$$

- We get

$$\begin{aligned} F(z) &= \left(\sum_{k=0}^{b-1} a(k)z^k\right) + \left(\sum_{k=0}^{b-1} a(k)z^k\right) z^b + \left(\sum_{k=0}^{b-1} a(k)z^k\right) z^{2b} + \cdots \\ &= \frac{\sum_{k=0}^{b-1} a(k)\, z^k}{1-z^b} = \frac{P(z)}{1-z^b}, \end{aligned}$$

▸ Write down the def of $P(z)$ on the board

where the last step simply defines $P(z) := \sum_{k=0}^{b-1} a(k)\, z^k$

Toward the general theory (2)

- By partial fraction expansion, we get

$$\begin{aligned} F(z) &= \frac{P(z)}{1-z^b} = \sum_{m=0}^{b-1} \frac{\hat{a}(m)}{1-\xi^m z} \\ &= \sum_{m=0}^{b-1} \hat{a}(m) \sum_{n\geq 0} \xi^{mn} z^n = \sum_{n\geq 0} \left(\sum_{m=0}^{b-1} \hat{a}(m)\, \xi^{mn}\right) z^n \end{aligned}$$

- By comparing the coefficients we get

$$a(n) = \sum_{m=0}^{b-1} \hat{a}(m)\, \xi^{mn}$$

Toward the general theory (3): Computing the Fourier coefficients

- We have

$$P(z) = \sum_{m=0}^{b-1} \hat{a}(m) \frac{1-z^b}{1-\xi^m z}$$

- It follows $P(\xi^{-n}) = b\,\hat{a}(n)$ since

$$m-n \not\equiv 0 \bmod b \Rightarrow \lim_{z\to\xi^{-n}} \frac{1-z^b}{1-\xi^m z} = 0$$

$$m-n \equiv 0 \bmod b \Rightarrow \lim_{z\to\xi^{-n}} \frac{1-z^b}{1-\xi^m z} = \lim_{z\to\xi^{-n}} \frac{bz^{b-1}}{\xi^m} = b\,\xi^{n-m} = b$$

- Therefore

$$\hat{a}(n) = \frac{1}{b} P(\xi^{-n}) = \frac{1}{b} \sum_{k=0}^{b-1} a(k)\, \xi^{-nk}$$

Main theorem (and Definition)

Theorem 7.2

$a(n)$ any periodic function on $\mathbb{Z}$ with period $b \Rightarrow$

$$a(n) = \sum_{k=0}^{b-1} \hat{a}(k)\, \xi^{nk},$$

where the Fourier coefficients are

$$\hat{a}(n) = \frac{1}{b} \sum_{k=0}^{b-1} a(k)\, \xi^{-nk},$$

with $\xi = e^{2\pi i/b}$ □

In this way, we obtain the general term of a periodic function

Matrix view of the finite Fourier series

The Fourier transform can be rephrased with a matrix:

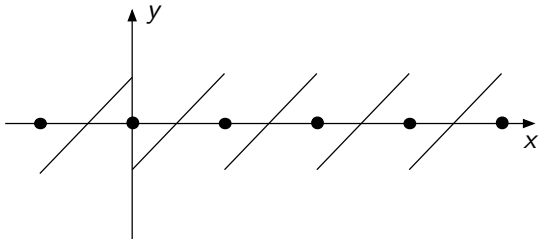
$$\begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \\ a(b-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{b-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{2(b-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & \xi^{b-1} & \xi^{2(b-1)} & \cdots & \xi^{(b-1)^2} \end{pmatrix} \begin{pmatrix} \hat{a}(0) \\ \hat{a}(1) \\ \hat{a}(2) \\ \vdots \\ \hat{a}(b-1) \end{pmatrix}$$

Example: Sawtooth functions

Definition (Sawtooth function)

The sawtooth function $((\cdot)) : \mathbb{R} \to \mathbb{R}$ is defined as

$$((x)) := \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$



Remark: $((x))$ is a periodic function with period 1

The finite Fourier series of sawtooth functions

Lemma 7.3

The finite Fourier series for the discrete sawtooth function $\left(\left(\frac{a}{b}\right)\right)$, a periodic function of $a \in \mathbb{Z}$ with period b, is given by

$$\left(\left(\frac{a}{b}\right)\right) = \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1+\xi^k}{1-\xi^k}\, \xi^{ak} = \frac{i}{2b} \sum_{k=1}^{b-1} \cot \frac{\pi k}{b}\, \xi^{ak}$$

Reminder

The cotangent of x is defined by $\cot x = \dfrac{\cos x}{\sin x}$

- It follows $\dfrac{1+e^{2\pi i x}}{1-e^{2\pi i x}} = i \cot(\pi x)$

Proof of Lemma 7.3

- $\left(\left(\frac{a}{b}\right)\right) = \sum_{k=0}^{b-1} \hat{a}(k)\,\xi^{ak}$ and $\hat{a}(k) = \frac{1}{b}\sum_{m=0}^{b-1}\left(\left(\frac{m}{b}\right)\right)\xi^{-mk}$ (Thm 7.2)
- $\hat{a}(0) = \frac{1}{b}\sum_{m=0}^{b-1}\left(\left(\frac{m}{b}\right)\right) = 0$ (Exer 7.14)
- When $k \neq 0$

$$\begin{aligned}\hat{a}(k) &= \frac{1}{b}\sum_{m=1}^{b-1}\left(\frac{m}{b}-\frac{1}{2}\right)\xi^{-mk} = \frac{1}{b^2}\sum_{m=1}^{b-1} m\,\xi^{-mk} + \frac{1}{2b} \\ &= \frac{1}{b}\left(\frac{\xi^k}{1-\xi^k}+\frac{1}{2}\right) \qquad \text{(Exer 7.5)} \\ &= \frac{1}{2b}\frac{1+\xi^k}{1-\xi^k} \qquad \square\end{aligned}$$

Another example: Dedekind sums

Definition (Dedekind sum)

For any two relatively prime integers a and $b > 0$, the Dedekind sum is defined by

$$s(a,b) = \sum_{k=0}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

Note: $s(a,b)$ is a periodic fn of the variable a with period b, i.e.,

- $s(a+jb,b) = s(a,b)$ for all $j \in \mathbb{Z}$
- $s(a,b) = s(a \bmod b, b)$

The finite Fourier series of Dedekind sums

Lemma 7.4

$$s(a,b) = \frac{1}{4b}\sum_{\mu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\,\frac{1+\xi^{-\mu a}}{1-\xi^{-\mu a}} = \frac{1}{4b}\sum_{\mu=1}^{b-1}\cot\frac{\pi\mu}{b}\cot\frac{\pi\mu a}{b}$$

Proof:

$$\begin{aligned}s(a,b) &= \sum_{k=0}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) \\ &= \frac{1}{4b^2}\sum_{k=0}^{b-1}\left(\left(\sum_{\mu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\xi^{\mu ka}\right)\left(\sum_{\nu=1}^{b-1}\frac{1+\xi^{\nu}}{1-\xi^{\nu}}\xi^{\nu k}\right)\right) \\ &= \frac{1}{4b^2}\sum_{\mu=1}^{b-1}\sum_{\nu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\frac{1+\xi^{\nu}}{1-\xi^{\nu}}\left(\sum_{k=0}^{b-1}\xi^{k(\nu+\mu a)}\right)\end{aligned}$$

Proof of Lemma 7.4 (cont'd)

- $s(a,b) = \frac{1}{4b^2}\sum_{\mu=1}^{b-1}\sum_{\nu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\frac{1+\xi^{\nu}}{1-\xi^{\nu}}\left(\sum_{k=0}^{b-1}\xi^{k(\nu+\mu a)}\right)$
- $\nu \not\equiv -\mu a \bmod b \Rightarrow \sum_{k=0}^{b-1}\xi^{k(\nu+\mu a)} = 0$ (Exer 7.6)
- $\nu \equiv -\mu a \bmod b \Rightarrow \sum_{k=0}^{b-1}\xi^{k(\nu+\mu a)} = b$
- $\therefore s(a,b) = \frac{1}{4b}\sum_{\mu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\frac{1+\xi^{-\mu a}}{1-\xi^{-\mu a}}$
- $s(a,b) = \frac{i^2}{4b}\sum_{\mu=1}^{b-1}\cot\frac{\pi\mu}{b}\cot\frac{-\pi\mu a}{b} = \frac{1}{4b}\sum_{\mu=1}^{b-1}\cot\frac{\pi\mu}{b}\cot\frac{\pi\mu a}{b}$ $\square$

1. A Motivating Example
2. Finite Fourier Series for Periodic Functions on $\mathbb{Z}$
3. The Finite Fourier Transform and Its Properties
4. The Parseval Identity
5. The Convolution of Finite Fourier Series

The Fourier transform

- f a given periodic function f on $\mathbb{Z}$ with period b
- f has a finite Fourier series, with coeff's $\hat{f}(0), \hat{f}(1), \ldots, \hat{f}(b-1)$
- f regarded as a function on the finite set $G = \{0, 1, 2, \ldots, b-1\}$
- $V_G :=$ the vect space of all complex-valued fn's on G
- (Or, $V_G :=$ the vect space of all periodic fns on $\mathbb{Z}$ with period b)

Definition (Fourier transform)

The Fourier transform of f, denoted by $\mathbf{F}(f)$, is the periodic function on $\mathbb{Z}$ defined by the sequence of uniquely determined values

$$\hat{f}(0),\ \hat{f}(1),\ \ldots,\ \hat{f}(b-1)\,;$$

Thus $\mathbf{F}(f)(m) = \hat{f}(m)$

Remark: $\mathbf{F}(f)$ is a linear transformation on V_G (Thm 7.2)

A basis of V_G

- Observation: $\dim V_G = b$
- In fact, the following functions form a basis of V_G (Exer 7.7)

$$\delta_m(x) := \begin{cases} 1 & \text{if } x = m + kb, \text{ for some integer } k, \\ 0 & \text{otherwise} \end{cases}$$

Another basis of V_G

- For any fixed integer a
  the roots of unity $\{\mathbf{e}_a(x) := e^{2\pi i a x/b} : x \in \mathbb{Z}\}$ can be thought of as a single function $\mathbf{e}_a \in V_G$
- The functions $\{\mathbf{e}_1, \ldots, \mathbf{e}_b\}$ give a basis of V_G (Thm 7.2)
- The relation of these two bases:

$$\widehat{\delta_a}(n) = \frac{1}{b}\, e^{-2\pi i a n/b},$$

since

$$\widehat{\delta_a}(n) = \frac{1}{b} \sum_{k=0}^{b-1} \delta_a(k)\, \xi^{-kn} = \frac{1}{b}\, \xi^{-an} = \frac{1}{b}\, e^{-2\pi i a n/b}$$

- Conclusion: the finite Fourier transform is the basis transform from the first to the second

Inner products on V_G

- Define the inner product of two functions $f, g \in V_G$ as

$$\langle f, g \rangle = \sum_{k=0}^{b-1} f(k)\overline{g(k)}$$

- It is actually an inner product (Exer 7.8)
  1. $\langle f, f \rangle \geq 0$, with equality if and only if $f = 0$
  2. $\langle f, g \rangle = \overline{\langle g, f \rangle}$
- Define the distance of two functions $f, g \in V_G$ as

$$d(f, g) = \langle f - g, f - g \rangle$$

Orthogonality relations

Lemma 7.5 (Orthogonality relations)

$$\frac{1}{b}\langle \mathbf{e}_a, \mathbf{e}_c \rangle = \delta_a(c) = \begin{cases} 1 & \text{if } b \mid (a-c), \\ 0 & \text{otherwise} \end{cases}$$

Proof: Just compute

$$\langle \mathbf{e}_a, \mathbf{e}_c \rangle = \sum_{m=0}^{b-1} \mathbf{e}_a(m)\overline{\mathbf{e}_c(m)} = \sum_{m=0}^{b-1} e^{2\pi i(a-c)m/b}$$

- $b \mid (a-c) \Rightarrow$ each term $= 1$
- $b \nmid (a-c) \Rightarrow e^{2\pi i m(a-c)/b}$ is a nontrivial root of unity, and then

$$\sum_{m=0}^{b-1} e^{\frac{2\pi i(a-c)m}{b}} = \frac{e^{b\frac{2\pi i(a-c)}{b}} - 1}{e^{\frac{2\pi i(a-c)}{b}} - 1} = 0 \qquad \square$$

Example 7.6

- Consider

$$B(k) := \left(\left(\frac{k}{b}\right)\right) = \begin{cases} \left\{\frac{k}{b}\right\} - \frac{1}{2} & \text{if } \frac{k}{b} \notin \mathbb{Z}, \\ 0 & \text{if } \frac{k}{b} \in \mathbb{Z} \end{cases}$$

- From the proof of Lemma 7.3, we have

$$\widehat{B}(n) = \frac{1}{2b}\frac{1+\xi^n}{1-\xi^n} = \frac{i}{2b}\cot\frac{\pi n}{b}$$

  for $n \neq 0$, and $\widehat{B}(0) = 0$

We'll use it in Example 7.9

1. A Motivating Example
2. Finite Fourier Series for Periodic Functions on $\mathbb{Z}$
3. The Finite Fourier Transform and Its Properties
4. The Parseval Identity
5. The Convolution of Finite Fourier Series

Parseval identity

Theorem 7.7 (Parseval identity)

For all $f \in V_G$,

$$\langle f, f \rangle = b \langle \hat{f}, \hat{f} \rangle$$

Proof:

- $\mathbf{e}_m(x) = e^{2\pi i m x/b} = \xi^{mx}$ (by def)
- By Thm 7.2

$$\hat{f}(x) = \frac{1}{b}\sum_{m=0}^{b-1} f(m)\,\xi^{-mx} = \frac{1}{b}\sum_{m=0}^{b-1} f(m)\,\overline{\mathbf{e}_m(x)}$$

Proof of Parseval indentity (cont'd)

$$\begin{aligned}
\langle \hat{f}, \hat{f} \rangle &= \left\langle \frac{1}{b}\sum_{m=0}^{b-1} f(m)\,\overline{\mathbf{e}_m},\ \frac{1}{b}\sum_{n=0}^{b-1} f(n)\,\overline{\mathbf{e}_n} \right\rangle && \text{(Thm 7.2)}\\
&= \frac{1}{b^2}\sum_{k=0}^{b-1}\sum_{m=0}^{b-1} f(m)\,\overline{\mathbf{e}_m(k)} \sum_{n=0}^{b-1} \overline{f(n)}\,\mathbf{e}_n(k)\\
&= \frac{1}{b^2}\sum_{m=0}^{b-1}\sum_{n=0}^{b-1} f(m)\,\overline{f(n)}\,\langle \mathbf{e}_m, \mathbf{e}_n \rangle\\
&= \frac{1}{b}\sum_{m=0}^{b-1}\sum_{n=0}^{b-1} f(m)\,\overline{f(n)}\,\delta_m(n) && \text{(Orthogonality)}\\
&= \frac{1}{b}\langle f, f \rangle && \square
\end{aligned}$$

A more general version

Theorem 7.8

For all $f, g \in V_G$, we have

$$\langle f, g \rangle = b \langle \hat{f}, \hat{g} \rangle$$

Proof: Similar to Theorem 7.7

Example 7.9: Reprove Lemma 7.4

Definition (Dedekind sum (recap))

For any two relatively prime integers a and $b > 0$, the Dedekind sum is defined by

$$s(a, b) = \sum_{k=0}^{b-1} \left(\!\!\left(\frac{ka}{b}\right)\!\!\right)\left(\!\!\left(\frac{k}{b}\right)\!\!\right)$$

Lemma 7.4 (recap)

For any two relatively prime integers a and $b > 0$,

$$s(a, b) = \frac{1}{4b}\sum_{\mu=1}^{b-1} \frac{1+\xi^{\mu}}{1-\xi^{\mu}}\,\frac{1+\xi^{-\mu a}}{1-\xi^{-\mu a}} = \frac{1}{4b}\sum_{\mu=1}^{b-1} \cot\frac{\pi\mu}{b}\cot\frac{\pi\mu a}{b}$$

Another proof: Based on Parseval identity

Example 7.9: The Fourier transforms

- Let $f(k) = \left(\left(\frac{k}{b}\right)\right)$ and $g(k) = \left(\left(\frac{ka}{b}\right)\right)$
- $\hat{f}(n) = \dfrac{i}{2b} \cot \dfrac{\pi n}{b}$ (Example 7.6)
- By Lem 7.3, $\left(\left(\dfrac{ka}{b}\right)\right) = \dfrac{i}{2b} \displaystyle\sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \xi^{mka}$
- By Exercise 1.9

$$\sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \xi^{mka} = \sum_{m=1}^{b-1} \cot \frac{\pi m a^{-1}}{b} \xi^{ma^{-1}ka} = \sum_{m=1}^{b-1} \cot \frac{\pi m a^{-1}}{b} \xi^{mk}$$

- Therefore $\hat{g}(n) = \dfrac{i}{2b} \cot \dfrac{\pi n a^{-1}}{b}$

Example 7.9: The Fourier series of Dedekind sums

Then, by Parseval identity,

$$\begin{aligned} s(a,b) &:= \sum_{k=0}^{b-1} \left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ka}{b}\right)\right) \\ &= b \sum_{m=1}^{b-1} \left(\frac{i}{2b} \cot \frac{\pi m}{b}\right) \overline{\left(\frac{i}{2b} \cot \frac{\pi m a^{-1}}{b}\right)} \\ &= \frac{1}{4b} \sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \cot \frac{\pi m a^{-1}}{b} \\ &= \frac{1}{4b} \sum_{m=1}^{b-1} \cot \frac{\pi m a}{b} \cot \frac{\pi m}{b} \end{aligned}$$

□

1. A Motivating Example
2. Finite Fourier Series for Periodic Functions on $\mathbb{Z}$
3. The Finite Fourier Transform and Its Properties
4. The Parseval Identity
5. The Convolution of Finite Fourier Series

The convolution of two periodic functions

Definition (Convolution)

For any complex-valued periodic functions f, g with period b, the convolution is

$$(f * g)(t) = \sum_{m=0}^{b-1} f(t-m) g(m)$$

Theorem 7.10 (Convolution theorem for finite Fourier series)

$f(t) = \dfrac{1}{b} \displaystyle\sum_{k=0}^{b-1} a_k \xi^{kt}$ and $g(t) = \dfrac{1}{b} \displaystyle\sum_{k=0}^{b-1} c_k \xi^{kt}$, where $\xi = e^{2\pi i/b}$

$$\Rightarrow \qquad (f * g)(t) = \frac{1}{b} \sum_{k=0}^{b-1} a_k c_k \xi^{kt}$$

Proof of Theorem 7.10

$$\begin{aligned}\sum_{m=0}^{b-1} f(t-m)g(m) &= \frac{1}{b^2}\sum_{m=0}^{b-1}\left(\sum_{k=0}^{b-1} a_k\,\xi^{k(t-m)}\right)\left(\sum_{l=0}^{b-1} c_l\,\xi^{lm}\right)\\ &= \frac{1}{b^2}\sum_{k=0}^{b-1}\sum_{l=0}^{b-1} a_k c_l\left(\sum_{m=0}^{b-1}\xi^{kt+(l-k)m}\right)\\ &= \frac{1}{b}\sum_{k=0}^{b-1} a_k c_k\,\xi^{kt}\end{aligned}$$

Note

- $l \neq k \Rightarrow \sum_{m=0}^{b-1}\xi^{(l-k)m} = 0$ (Exer 7.6)
- $l = k \Rightarrow \sum_{m=0}^{b-1}\xi^{(l-k)m} = b$ $\square$

Example 7.11

Goal

Prove
$$\sum_{k=1}^{b-1}\cot^2\left(\frac{\pi k}{b}\right) = \frac{(b-1)(b-2)}{3}$$

Proof: Use the convolution

- Let $f(t) = g(t) = \frac{1}{b}\sum_{k=1}^{b-1}\cot\frac{\pi k}{b}\,\xi^{kt}$
- Then $(f * g)(t) = \frac{1}{b}\sum_{k=1}^{b-1}\cot^2\frac{\pi k}{b}\,\xi^{kt}$
- Didn't we see the fnct $\frac{1}{b}\sum_{k=1}^{b-1}\cot\frac{\pi k}{b}\,\xi^{kt}$ somewhere before?

Example 7.11 (cont'd)

- By Lem 7.3 $\left(\!\left(\frac{t}{b}\right)\!\right) = \frac{i}{2b}\sum_{k=1}^{b-1}\cot\frac{\pi k}{b}\,\xi^{kt}$
- $\therefore f(t) = g(t) = \frac{2}{i}\left(\!\left(\frac{t}{b}\right)\!\right)$
- $\therefore (f * g)(t) = -4\sum_{m=1}^{b-1}\left(\!\left(\frac{t-m}{b}\right)\!\right)\left(\!\left(\frac{m}{b}\right)\!\right)$ (by def)
- Hence

$$-4\sum_{m=1}^{b-1}\left(\!\left(\frac{t-m}{b}\right)\!\right)\left(\!\left(\frac{m}{b}\right)\!\right) = \frac{1}{b}\sum_{k=1}^{b-1}\cot^2\frac{\pi k}{b}\,\xi^{kt}$$

$$\therefore\quad -4b\sum_{m=1}^{b-1}\left(\!\left(\frac{t-m}{b}\right)\!\right)\left(\!\left(\frac{m}{b}\right)\!\right) = \sum_{k=1}^{b-1}\cot^2\frac{\pi k}{b}\,\xi^{kt}$$

Example 7.11 (further cont'd)

- Setting $t = 0$, we get

$$-4b\sum_{m=1}^{b-1}\left(\!\left(\frac{-m}{b}\right)\!\right)\left(\!\left(\frac{m}{b}\right)\!\right) = \sum_{k=1}^{b-1}\cot^2\frac{\pi k}{b}$$

- Furthermore

$$\begin{aligned}-4b\sum_{m=1}^{b-1}\left(\!\left(\frac{-m}{b}\right)\!\right)\left(\!\left(\frac{m}{b}\right)\!\right) &= 4b\sum_{m=1}^{b-1}\left(\!\left(\frac{m}{b}\right)\!\right)\left(\!\left(\frac{m}{b}\right)\!\right)\\ &= 4b\sum_{m=1}^{b-1}\left(\left\{\frac{m}{b}\right\}-\frac{1}{2}\right)^2\\ &= 4b\sum_{m=1}^{b-1}\left(\frac{m}{b}-\frac{1}{2}\right)^2\\ &= \frac{(b-1)(b-2)}{3}\end{aligned}$$

$\square$

The goal of this chapter

Motivation

- Quasipolynomials involve periodic functions
- Especially Fourier–Dedekind sums

The goal of this chapter

1. Establish a theory for periodic functions
2. Prepare for the next chapter