Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(8) Finite Fourier Analysis

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Important theorems from the previous lectures

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the denominator of $\mathcal{P}$

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

The goal of this chapter

Motivation

- Quasipolynomials involve periodic functions
- Especially Fourier–Dedekind sums

The goal of this chapter

1. Establish a theory for periodic functions
2. Prepare for the next chapter

Example 7.1

- Consider the following function with period 3

$$
\begin{aligned}
n &: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots \\
a(n) &: 1, 5, 2, 1, 5, 2, 1, 5, 2, 1, \ldots
\end{aligned}
$$

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- Embed this sequence into a generating function as follows

$$F(z) := 1 + 5z + 2z^2 + z^3 + 5z^4 + 2z^5 + \cdots = \sum_{n \geq 0} a(n)\, z^n$$

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- Then

$$F(z) = 1+5z+2z^2+z^3\left(1+5z+2z^2\right)+z^6\left(1+5z+2z^2\right)+\cdots$$

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$$\begin{aligned} F(z) &= 1+5z+2z^2+z^3\left(1+5z+2z^2\right)+z^6\left(1+5z+2z^2\right)+\cdots \\ &= \left(1+5z+2z^2\right)\sum_{k \geq 0} z^{3k} \end{aligned}$$

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$$\begin{aligned} F(z) &= 1{+}5z{+}2z^2{+}z^3\left(1{+}5z{+}2z^2\right) {+}z^6\left(1{+}5z{+}2z^2\right) + \cdots \\ &= \left(1{+}5z{+}2z^2\right) \sum_{k \geq 0} z^{3k} = \frac{1 + 5z + 2z^2}{1 - z^3} \end{aligned}$$

Example 7.1 (cont'd)

- By the partial fraction expansion we obtain

$$F(z) = \frac{\hat{a}(0)}{1-z} + \frac{\hat{a}(1)}{1-\rho z} + \frac{\hat{a}(2)}{1-\rho^2 z}$$

  - the constants $\hat{a}(0), \hat{a}(1), \hat{a}(2)$ remain to be found
  - $\rho := e^{2\pi i/3}$

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$$F(z) = \hat{a}(0)\sum_{n\geq 0} z^n + \hat{a}(1)\sum_{n\geq 0}(\rho z)^n + \hat{a}(2)\sum_{n\geq 0}(\rho^2 z)^n$$

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- Then

$$\begin{aligned} F(z) &= \hat{a}(0) \sum_{n \geq 0} z^n + \hat{a}(1) \sum_{n \geq 0} (\rho z)^n + \hat{a}(2) \sum_{n \geq 0} (\rho^2 z)^n \\ &= \sum_{n \geq 0} \left(\hat{a}(0) + \hat{a}(1)\rho^n + \hat{a}(2)\rho^{2n} \right) z^n \end{aligned}$$

We've derived the finite Fourier series of our sequence $a(n)$!

Example 7.1 (further cont'd)

- To obtain $\hat{a}(0), \hat{a}(1), \hat{a}(2)$, we use the identify for all $z \in \mathbb{C}$

$$
\begin{aligned}
1 + 5z + 2z^2 = \hat{a}(0)\,(1-\rho z)\left(1-\rho^2 z\right) + \hat{a}(1)\,(1-z)\left(1-\rho^2 z\right) \\
+ \hat{a}(2)\,(1-z)\,(1-\rho z)
\end{aligned}
$$

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- By letting $z = 1,\ \rho^2,\ \rho$, respectively, we obtain

$$\begin{aligned} 1+5+2 &= 3\,\hat{a}(0) \\ 1+5\rho^2+2\rho^4 &= 3\,\hat{a}(1) \\ 1+5\rho+2\rho^2 &= 3\,\hat{a}(2) \end{aligned}$$

where we used the identity $(1-\rho)(1-\rho^2) = 3$ (Exer 7.2)

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- Then, we get the finite Fourier series for our sequence

$$a(n) = \frac{8}{3} + \left(-\frac{4}{3} - \rho\right)\rho^n + \left(-\frac{1}{3} + \rho\right)\rho^{2n} \qquad \square$$

Toward the general theory (1)

- $\{a(n)\}_{n=0}^{\infty}$ any periodic sequence on $\mathbb{Z}$ of period b
- Throughout the chapter, we fix $\xi := e^{2\pi i/b}$

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$$= \frac{\sum_{k=0}^{b-1} a(k)\, z^k}{1 - z^b} = \frac{P(z)}{1 - z^b},$$

▸ Write down the def of $P(z)$ on the board

where the last step simply defines $P(z) := \sum_{k=0}^{b-1} a(k)\, z^k$

Toward the general theory (2)

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- By comparing the coefficients we get

$$a(n) = \sum_{m=0}^{b-1} \hat{a}(m)\, \xi^{mn}$$

Toward the general theory (3): Computing the Fourier coefficients

- We have

$$P(z) = \sum_{m=0}^{b-1} \hat{a}(m) \frac{1 - z^b}{1 - \xi^m z}$$

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- It follows $P(\xi^{-n}) = b\,\hat{a}(n)$ since

$$m - n \not\equiv 0 \mod b \Rightarrow \lim_{z \to \xi^{-n}} \frac{1 - z^b}{1 - \xi^m z} = 0$$

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- Therefore
$$\hat{a}(n) = \frac{1}{b} P(\xi^{-n})$$

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- Therefore

$$\hat{a}(n) = \frac{1}{b} P(\xi^{-n}) = \frac{1}{b} \sum_{k=0}^{b-1} a(k)\,\xi^{-nk}$$

Main theorem (and Definition)

Theorem 7.2

$a(n)$ any periodic function on $\mathbb{Z}$ with period $b \Rightarrow$

$$a(n) = \sum_{k=0}^{b-1} \hat{a}(k)\,\xi^{nk},$$

where the Fourier coefficients are

$$\hat{a}(n) = \frac{1}{b}\sum_{k=0}^{b-1} a(k)\,\xi^{-nk},$$

with $\xi = e^{2\pi i/b}$ □

In this way, we obtain the general term of a periodic function

Matrix view of the finite Fourier series

The Fourier transform can be rephrased with a matrix:

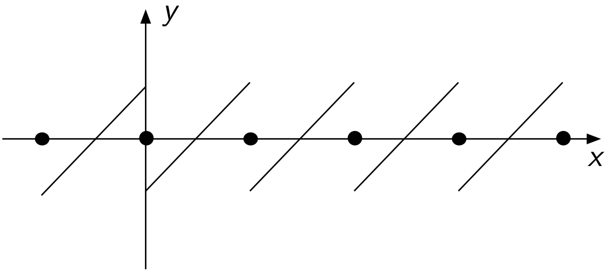
$$\begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \\ a(b-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{b-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{2(b-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & \xi^{b-1} & \xi^{2(b-1)} & \cdots & \xi^{(b-1)^2} \end{pmatrix} \begin{pmatrix} \hat{a}(0) \\ \hat{a}(1) \\ \hat{a}(2) \\ \vdots \\ \hat{a}(b-1) \end{pmatrix}$$

Example: Sawtooth functions

Definition (Sawtooth function)

The sawtooth function $((\cdot)) : \mathbb{R} \to \mathbb{R}$ is defined as

$$((x)) := \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$



Remark: $((x))$ is a periodic function with period 1

The finite Fourier series of sawtooth functions

Lemma 7.3

The finite Fourier series for the discrete sawtooth function $\left(\left(\frac{a}{b}\right)\right)$, a periodic function of $a \in \mathbb{Z}$ with period b, is given by

$$\left(\left(\frac{a}{b}\right)\right) = \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1+\xi^k}{1-\xi^k}\,\xi^{ak}$$

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$$\left(\left(\frac{a}{b}\right)\right) = \frac{1}{2b}\sum_{k=1}^{b-1}\frac{1+\xi^k}{1-\xi^k}\xi^{ak} = \frac{i}{2b}\sum_{k=1}^{b-1}\cot\frac{\pi k}{b}\xi^{ak}$$

Reminder

The cotangent of x is defined by $\cot x = \dfrac{\cos x}{\sin x}$

- It follows $\dfrac{1+e^{2\pi ix}}{1-e^{2\pi ix}} = i\cot(\pi x)$

Proof of Lemma 7.3

- $\left(\left(\frac{a}{b}\right)\right) = \sum_{k=0}^{b-1} \hat{a}(k)\, \xi^{ak}$ and $\hat{a}(k) = \frac{1}{b} \sum_{m=0}^{b-1} \left(\left(\frac{m}{b}\right)\right) \xi^{-mk}$

(Thm 7.2)

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- $$\left(\left(\frac{a}{b}\right)\right) = \sum_{k=0}^{b-1} \hat{a}(k)\,\xi^{ak} \text{ and } \hat{a}(k) = \frac{1}{b}\sum_{m=0}^{b-1}\left(\left(\frac{m}{b}\right)\right)\xi^{-mk}$$
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- $$\hat{a}(0) = \frac{1}{b}\sum_{m=0}^{b-1}\left(\left(\frac{m}{b}\right)\right) = 0$$
(Exer 7.14)

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- $\hat{a}(0) = \frac{1}{b}\sum_{m=0}^{b-1} \left(\left(\frac{m}{b}\right)\right) = 0$ (Exer 7.14)
- When $k \neq 0$

$$\hat{a}(k) = \frac{1}{b}\sum_{m=1}^{b-1} \left(\frac{m}{b} - \frac{1}{2}\right)\xi^{-mk}$$

□

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$$= \frac{1}{b}\left(\frac{\xi^k}{1-\xi^k} + \frac{1}{2}\right)$$ (Exer 7.5)

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$$= \frac{1}{b}\left(\frac{\xi^k}{1-\xi^k} + \frac{1}{2}\right) \quad \text{(Exer 7.5)}$$

$$= \frac{1}{2b}\frac{1+\xi^k}{1-\xi^k}$$

□

Another example: Dedekind sums

Definition (Dedekind sum)

For any two relatively prime integers a and $b > 0$, the Dedekind sum is defined by

$$s(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right) \left(\left(\frac{k}{b}\right)\right)$$

Note: $s(a,b)$ is a periodic fn of the variable a with period b, i.e.,

- $s(a + jb, b) = s(a,b)$ for all $j \in \mathbb{Z}$
- $s(a,b) = s(a \bmod b, b)$

The finite Fourier series of Dedekind sums

Lemma 7.4

$$s(a,b) = \frac{1}{4b}\sum_{\mu=1}^{b-1} \frac{1+\xi^{\mu}}{1-\xi^{\mu}}\,\frac{1+\xi^{-\mu a}}{1-\xi^{-\mu a}} = \frac{1}{4b}\sum_{\mu=1}^{b-1} \cot\frac{\pi\mu}{b}\cot\frac{\pi\mu a}{b}$$

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Proof:

$$s(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

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Proof:

$$s(a,b) = \sum_{k=0}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

$$= \frac{1}{4b^2}\sum_{k=0}^{b-1}\left(\left(\sum_{\mu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\xi^{\mu k a}\right)\left(\sum_{\nu=1}^{b-1}\frac{1+\xi^{\nu}}{1-\xi^{\nu}}\xi^{\nu k}\right)\right)$$

The finite Fourier series of Dedekind sums

Lemma 7.4

$$s(a,b) = \frac{1}{4b}\sum_{\mu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\,\frac{1+\xi^{-\mu a}}{1-\xi^{-\mu a}} = \frac{1}{4b}\sum_{\mu=1}^{b-1}\cot\frac{\pi\mu}{b}\cot\frac{\pi\mu a}{b}$$

Proof:

$$\begin{aligned} s(a,b) &= \sum_{k=0}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) \\ &= \frac{1}{4b^2}\sum_{k=0}^{b-1}\left(\left(\sum_{\mu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\xi^{\mu ka}\right)\left(\sum_{\nu=1}^{b-1}\frac{1+\xi^{\nu}}{1-\xi^{\nu}}\xi^{\nu k}\right)\right) \\ &= \frac{1}{4b^2}\sum_{\mu=1}^{b-1}\sum_{\nu=1}^{b-1}\frac{1+\xi^{\mu}}{1-\xi^{\mu}}\frac{1+\xi^{\nu}}{1-\xi^{\nu}}\left(\sum_{k=0}^{b-1}\xi^{k(\nu+\mu a)}\right) \end{aligned}$$

Proof of Lemma 7.4 (cont'd)

- $s(a, b) = \frac{1}{4b^2} \sum_{\mu=1}^{b-1} \sum_{\nu=1}^{b-1} \frac{1+\xi^\mu}{1-\xi^\mu} \frac{1+\xi^\nu}{1-\xi^\nu} \left(\sum_{k=0}^{b-1} \xi^{k(\nu+\mu a)} \right)$

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Proof of Lemma 7.4 (cont'd)

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- $s(a,b) = \frac{i^2}{4b}\sum_{\mu=1}^{b-1}\cot\frac{\pi\mu}{b}\cot\frac{-\pi\mu a}{b}$

Proof of Lemma 7.4 (cont'd)

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The Fourier transform

- f a given periodic function f on $\mathbb{Z}$ with period b

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The Fourier transform of f, denoted by $\mathbf{F}(f)$, is the periodic function on $\mathbb{Z}$ defined by the sequence of uniquely determined values

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Thus $\mathbf{F}(f)(m) = \hat{f}(m)$

Remark: $\mathbf{F}(f)$ is a linear transformation on V_G (Thm 7.2)

A basis of V_G

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- In fact, the following functions form a basis of V_G (Exer 7.7)

$$\delta_m(x) := \begin{cases} 1 & \text{if } x = m + kb, \text{ for some integer } k, \\ 0 & \text{otherwise} \end{cases}$$

Another basis of V_G

- For any fixed integer a the roots of unity $\{\mathbf{e}_a(x) := e^{2\pi i a x/b} : x \in \mathbb{Z}\}$ can be thought of as a single function $\mathbf{e}_a \in V_G$

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- Conclusion: the finite Fourier transform is the basis transform from the first to the second

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  1. $\langle f, f \rangle \geq 0$, with equality if and only if $f = 0$
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- Define the distance of two functions $f, g \in V_G$ as

$$d(f, g) = \langle f - g, f - g \rangle$$

Orthogonality relations

Lemma 7.5 (Orthogonality relations)

$$\frac{1}{b}\langle \mathbf{e}_a, \mathbf{e}_c \rangle = \delta_a(c) = \begin{cases} 1 & \text{if } b \mid (a-c), \\ 0 & \text{otherwise} \end{cases}$$

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$$\sum_{m=0}^{b-1} e^{\frac{2\pi i(a-c)m}{b}} = \frac{e^{b\frac{2\pi i(a-c)}{b}} - 1}{e^{\frac{2\pi i(a-c)}{b}} - 1}$$

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Example 7.6

- Consider

$$B(k) := \left(\left(\frac{k}{b}\right)\right) = \begin{cases} \left\{\frac{k}{b}\right\} - \frac{1}{2} & \text{if } \frac{k}{b} \notin \mathbb{Z}, \\ 0 & \text{if } \frac{k}{b} \in \mathbb{Z} \end{cases}$$

We'll use it in Example 7.9

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- From the proof of Lemma 7.3, we have

$$\widehat{B}(n) = \frac{1}{2b}\frac{1+\xi^n}{1-\xi^n} = \frac{i}{2b}\cot\frac{\pi n}{b}$$

  for $n \neq 0$, and $\widehat{B}(0) = 0$

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Parseval identity

Theorem 7.7 (Parseval identity)

For all $f \in V_G$,

$$\langle f, f \rangle = b \langle \hat{f}, \hat{f} \rangle$$

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$$\hat{f}(x) = \frac{1}{b} \sum_{m=0}^{b-1} f(m)\, \xi^{-mx}$$

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$$\hat{f}(x) = \frac{1}{b} \sum_{m=0}^{b-1} f(m)\, \xi^{-mx} = \frac{1}{b} \sum_{m=0}^{b-1} f(m)\, \overline{\mathbf{e}_m(x)}$$

Proof of Parseval indentity (cont'd)

$$\langle \hat{f}, \hat{f} \rangle = \left\langle \; \frac{1}{b} \sum_{m=0}^{b-1} f(m)\, \overline{\mathbf{e}_m}\, , \; \frac{1}{b} \sum_{n=0}^{b-1} f(n)\, \overline{\mathbf{e}_n} \; \right\rangle \qquad \text{(Thm 7.2)}$$

□

Proof of Parseval indentity (cont'd)

$$\langle \hat{f}, \hat{f} \rangle = \left\langle \frac{1}{b} \sum_{m=0}^{b-1} f(m)\,\overline{\mathbf{e}_m}\,,\ \frac{1}{b} \sum_{n=0}^{b-1} f(n)\,\overline{\mathbf{e}_n} \right\rangle \qquad \text{(Thm 7.2)}$$

$$= \frac{1}{b^2} \sum_{k=0}^{b-1} \sum_{m=0}^{b-1} f(m)\,\overline{\mathbf{e}_m(k)} \sum_{n=0}^{b-1} \overline{f(n)}\,\mathbf{e}_n(k)$$

□

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$$= \frac{1}{b^2} \sum_{m=0}^{b-1} \sum_{n=0}^{b-1} f(m)\, \overline{f(n)}\, \langle \mathbf{e}_m, \mathbf{e}_n \rangle$$

□

Proof of Parseval indentity (cont'd)

$$
\begin{aligned}
\langle \hat{f}, \hat{f} \rangle &= \left\langle \frac{1}{b} \sum_{m=0}^{b-1} f(m)\, \overline{\mathbf{e}_m}\,,\ \frac{1}{b} \sum_{n=0}^{b-1} f(n)\, \overline{\mathbf{e}_n} \right\rangle && \text{(Thm 7.2)} \\
&= \frac{1}{b^2} \sum_{k=0}^{b-1} \sum_{m=0}^{b-1} f(m)\, \overline{\mathbf{e}_m(k)} \sum_{n=0}^{b-1} \overline{f(n)}\, \mathbf{e}_n(k) \\
&= \frac{1}{b^2} \sum_{m=0}^{b-1} \sum_{n=0}^{b-1} f(m)\, \overline{f(n)}\, \langle \mathbf{e}_m, \mathbf{e}_n \rangle \\
&= \frac{1}{b} \sum_{m=0}^{b-1} \sum_{n=0}^{b-1} f(m)\, \overline{f(n)}\, \delta_m(n) && \text{(Orthogonality)}
\end{aligned}
$$

□

Proof of Parseval indentity (cont'd)

$$\begin{aligned}
\langle \hat{f}, \hat{f} \rangle &= \left\langle \frac{1}{b} \sum_{m=0}^{b-1} f(m)\, \overline{\mathbf{e}_m},\ \frac{1}{b} \sum_{n=0}^{b-1} f(n)\, \overline{\mathbf{e}_n} \right\rangle && \text{(Thm 7.2)} \\
&= \frac{1}{b^2} \sum_{k=0}^{b-1} \sum_{m=0}^{b-1} f(m)\, \overline{\mathbf{e}_m(k)} \sum_{n=0}^{b-1} \overline{f(n)}\, \mathbf{e}_n(k) \\
&= \frac{1}{b^2} \sum_{m=0}^{b-1} \sum_{n=0}^{b-1} f(m)\, \overline{f(n)}\, \langle \mathbf{e}_m, \mathbf{e}_n \rangle \\
&= \frac{1}{b} \sum_{m=0}^{b-1} \sum_{n=0}^{b-1} f(m)\, \overline{f(n)}\, \delta_m(n) && \text{(Orthogonality)} \\
&= \frac{1}{b} \langle f, f \rangle && \square
\end{aligned}$$

A more general version

Theorem 7.8

For all $f, g \in V_G$, we have

$$\langle f, g \rangle = b \langle \hat{f}, \hat{g} \rangle$$

Proof: Similar to Theorem 7.7

Example 7.9: Reprove Lemma 7.4

Definition (Dedekind sum (recap))

For any two relatively prime integers a and $b > 0$, the Dedekind sum is defined by

$$s(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right) \left(\left(\frac{k}{b}\right)\right)$$

Lemma 7.4 (recap)

For any two relatively prime integers a and $b > 0$,

$$s(a,b) = \frac{1}{4b} \sum_{\mu=1}^{b-1} \frac{1+\xi^{\mu}}{1-\xi^{\mu}} \frac{1+\xi^{-\mu a}}{1-\xi^{-\mu a}} = \frac{1}{4b} \sum_{\mu=1}^{b-1} \cot \frac{\pi\mu}{b} \cot \frac{\pi\mu a}{b}$$

Another proof: Based on Parseval identity

Example 7.9: The Fourier transforms

- Let $f(k) = ((\frac{k}{b}))$ and $g(k) = ((\frac{ka}{b}))$
- $\hat{f}(n) = \dfrac{i}{2b} \cot \dfrac{\pi n}{b}$ (Example 7.6)
- By Lem 7.3, $\left(\left(\dfrac{ka}{b}\right)\right) = \dfrac{i}{2b} \sum_{m=1}^{b-1} \cot \dfrac{\pi m}{b} \xi^{mka}$

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- By Exercise 1.9

$$\sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \xi^{mka} = \sum_{m=1}^{b-1} \cot \frac{\pi m a^{-1}}{b} \xi^{m a^{-1} ka}$$

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- Let $f(k) = ((\frac{k}{b}))$ and $g(k) = ((\frac{ka}{b}))$
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- By Lem 7.3, $\left(\left(\frac{ka}{b}\right)\right) = \frac{i}{2b} \sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \xi^{mka}$
- By Exercise 1.9

$$\sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \xi^{mka} = \sum_{m=1}^{b-1} \cot \frac{\pi m a^{-1}}{b} \xi^{ma^{-1}ka} = \sum_{m=1}^{b-1} \cot \frac{\pi m a^{-1}}{b} \xi^{mk}$$

Example 7.9: The Fourier transforms

- Let $f(k) = ((\frac{k}{b}))$ and $g(k) = ((\frac{ka}{b}))$
- $\hat{f}(n) = \dfrac{i}{2b} \cot \dfrac{\pi n}{b}$ (Example 7.6)
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$$\sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \xi^{mka} = \sum_{m=1}^{b-1} \cot \frac{\pi m a^{-1}}{b} \xi^{ma^{-1}ka} = \sum_{m=1}^{b-1} \cot \frac{\pi m a^{-1}}{b} \xi^{mk}$$

- Therefore $\hat{g}(n) = \dfrac{i}{2b} \cot \dfrac{\pi n a^{-1}}{b}$

Example 7.9: The Fourier series of Dedekind sums

Then, by Parseval identity,

$$s(a,b) := \sum_{k=0}^{b-1} \left(\left(\frac{k}{b}\right)\right) \left(\left(\frac{ka}{b}\right)\right)$$

□

Example 7.9: The Fourier series of Dedekind sums

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$$
\begin{aligned}
s(a,b) &:= \sum_{k=0}^{b-1} \left(\left(\frac{k}{b}\right)\right) \left(\left(\frac{ka}{b}\right)\right) \\
&= b \sum_{m=1}^{b-1} \left(\frac{i}{2b} \cot \frac{\pi m}{b}\right) \overline{\left(\frac{i}{2b} \cot \frac{\pi m a^{-1}}{b}\right)}
\end{aligned}
$$

□

Example 7.9: The Fourier series of Dedekind sums

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$$
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s(a,b) &:= \sum_{k=0}^{b-1} \left(\left(\frac{k}{b}\right)\right) \left(\left(\frac{ka}{b}\right)\right) \\
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&= \frac{1}{4b} \sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \cot \frac{\pi m a^{-1}}{b}
\end{aligned}
$$

□

Example 7.9: The Fourier series of Dedekind sums

Then, by Parseval identity,

$$\begin{aligned} s(a,b) &:= \sum_{k=0}^{b-1} \left(\!\left(\frac{k}{b}\right)\!\right) \left(\!\left(\frac{ka}{b}\right)\!\right) \\ &= b \sum_{m=1}^{b-1} \left(\frac{i}{2b} \cot \frac{\pi m}{b}\right) \overline{\left(\frac{i}{2b} \cot \frac{\pi m a^{-1}}{b}\right)} \\ &= \frac{1}{4b} \sum_{m=1}^{b-1} \cot \frac{\pi m}{b} \cot \frac{\pi m a^{-1}}{b} \\ &= \frac{1}{4b} \sum_{m=1}^{b-1} \cot \frac{\pi m a}{b} \cot \frac{\pi m}{b} \end{aligned}$$

□

The convolution of two periodic functions

Definition (Convolution)

For any complex-valued periodic functions f, g with period b, the convolution is

$$(f * g)(t) = \sum_{m=0}^{b-1} f(t-m)g(m)$$

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Theorem 7.10 (Convolution theorem for finite Fourier series)

$$f(t) = \frac{1}{b}\sum_{k=0}^{b-1} a_k\,\xi^{kt} \text{ and } g(t) = \frac{1}{b}\sum_{k=0}^{b-1} c_k\,\xi^{kt}, \text{ where } \xi = e^{2\pi i/b}$$

$$\Rightarrow \qquad (f * g)(t) = \frac{1}{b}\sum_{k=0}^{b-1} a_k c_k\,\xi^{kt}$$

Proof of Theorem 7.10

$$\sum_{m=0}^{b-1} f(t-m)g(m) = \frac{1}{b^2}\sum_{m=0}^{b-1}\left(\sum_{k=0}^{b-1} a_k\,\xi^{k(t-m)}\right)\left(\sum_{l=0}^{b-1} c_l\,\xi^{lm}\right)$$

Proof of Theorem 7.10

$$\sum_{m=0}^{b-1} f(t-m)g(m) = \frac{1}{b^2}\sum_{m=0}^{b-1}\left(\sum_{k=0}^{b-1} a_k\,\xi^{k(t-m)}\right)\left(\sum_{l=0}^{b-1} c_l\,\xi^{lm}\right)$$
$$= \frac{1}{b^2}\sum_{k=0}^{b-1}\sum_{l=0}^{b-1} a_k c_l\left(\sum_{m=0}^{b-1}\xi^{kt+(l-k)m}\right)$$

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Note

- $l \neq k \Rightarrow \sum_{m=0}^{b-1}\xi^{(l-k)m} = 0$ (Exer 7.6)

Proof of Theorem 7.10

$$\sum_{m=0}^{b-1} f(t-m)g(m) = \frac{1}{b^2} \sum_{m=0}^{b-1} \left(\sum_{k=0}^{b-1} a_k\, \xi^{k(t-m)} \right) \left(\sum_{l=0}^{b-1} c_l\, \xi^{lm} \right)$$
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- $l \neq k \Rightarrow \sum_{m=0}^{b-1} \xi^{(l-k)m} = 0$ (Exer 7.6)
- $l = k \Rightarrow \sum_{m=0}^{b-1} \xi^{(l-k)m} = b$

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Example 7.11

Goal

Prove

$$\sum_{k=1}^{b-1} \cot^2\left(\frac{\pi k}{b}\right) = \frac{(b-1)(b-2)}{3}$$

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- Then $(f * g)(t) = \frac{1}{b}\sum_{k=1}^{b-1} \cot^2\frac{\pi k}{b}\,\xi^{kt}$

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- Then $(f * g)(t) = \frac{1}{b}\sum_{k=1}^{b-1} \cot^2\frac{\pi k}{b}\,\xi^{kt}$
- Didn't we see the fnct $\frac{1}{b}\sum_{k=1}^{b-1} \cot\frac{\pi k}{b}\,\xi^{kt}$ somewhere before?

Example 7.11 (cont'd)

- By Lem 7.3 $\left(\left(\frac{t}{b}\right)\right) = \frac{i}{2b}\sum_{k=1}^{b-1} \cot\frac{\pi k}{b}\,\xi^{kt}$

Example 7.11 (cont'd)

- By Lem 7.3 $\left(\left(\frac{t}{b}\right)\right) = \frac{i}{2b} \sum_{k=1}^{b-1} \cot \frac{\pi k}{b} \xi^{kt}$
- $\therefore f(t) = g(t) = \frac{2}{i} \left(\left(\frac{t}{b}\right)\right)$

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- By Lem 7.3 $\left(\left(\frac{t}{b}\right)\right) = \frac{i}{2b}\sum_{k=1}^{b-1} \cot\frac{\pi k}{b}\,\xi^{kt}$
- $\therefore f(t) = g(t) = \frac{2}{i}\left(\left(\frac{t}{b}\right)\right)$
- $\therefore (f * g)(t) = -4\sum_{m=1}^{b-1}\left(\left(\frac{t-m}{b}\right)\right)\left(\left(\frac{m}{b}\right)\right)$ (by def)

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- Hence

$$-4\sum_{m=1}^{b-1}\left(\left(\frac{t-m}{b}\right)\right)\left(\left(\frac{m}{b}\right)\right) = \frac{1}{b}\sum_{k=1}^{b-1}\cot^2\frac{\pi k}{b}\,\xi^{kt}$$

Example 7.11 (cont'd)

- By Lem 7.3 $\left(\left(\frac{t}{b}\right)\right) = \frac{i}{2b}\sum_{k=1}^{b-1} \cot\frac{\pi k}{b}\,\xi^{kt}$
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Example 7.11 (further cont'd)

- Setting $t = 0$, we get

$$-4b \sum_{m=1}^{b-1} \left(\left(\frac{-m}{b}\right)\right) \left(\left(\frac{m}{b}\right)\right) = \sum_{k=1}^{b-1} \cot^2 \frac{\pi k}{b}$$

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-4b \sum_{m=1}^{b-1} \left(\left(\frac{-m}{b}\right)\right) \left(\left(\frac{m}{b}\right)\right) &= 4b \sum_{m=1}^{b-1} \left(\left(\frac{m}{b}\right)\right) \left(\left(\frac{m}{b}\right)\right) \\
&= 4b \sum_{m=1}^{b-1} \left(\left\{\frac{m}{b}\right\} - \frac{1}{2}\right)^2 \\
&= 4b \sum_{m=1}^{b-1} \left(\frac{m}{b} - \frac{1}{2}\right)^2 \\
&= \frac{(b-1)(b-2)}{3}
\end{aligned}$$

The goal of this chapter

Motivation

- Quasipolynomials involve periodic functions
- Especially Fourier–Dedekind sums

The goal of this chapter

1. Establish a theory for periodic functions
2. Prepare for the next chapter