

Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(7) Magic squares

Yoshio Okamoto

Tokyo Institute of Technology

June 11, 2009

"Last updated: 2009/06/10 11:48"

Important theorems from the previous lectures

Theorem 3.8 (Ehrhart's Theorem)

\mathcal{P} is an integral convex d -polytope \Rightarrow
 $L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

\mathcal{P} is a rational convex d -polytope \Rightarrow
 $L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d ;
Its period divides the denominator of \mathcal{P}

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

\mathcal{P} a convex rational polytope \Rightarrow for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

- ① It's a Kind of Magic
- ② Semimagic Squares: Integer Points in the Birkhoff–von Neumann Polytope
- ③ Magic Generating Functions and Constant-Term Identities
- ④ The Enumeration of Magic Squares

The goal of this chapter

The goal of this chapter

- ① Count the number of magic squares by means of Ehrhart theory

A magic square in an engraving



Melencolia I
Albrecht Dürer (1514)

A magic square in architecture



Temple de la Sagrada Família
Barcelona, Spain

It's a Kind of Magic

① It's a Kind of Magic

② Semimagic Squares: Integer Points in the Birkhoff–von Neumann Polytope

③ Magic Generating Functions and Constant-Term Identities

④ The Enumeration of Magic Squares

It's a Kind of Magic

You may think these definitions are too weak...

Definition (Semimagic square, Magic square)

- A **semimagic square** is a square matrix whose entries are nonnegative integers and whose rows and columns sum to the same number (called the **line sum**)
 - The rows and columns are called **lines**
- A **magic square** is a semimagic square whose main diagonals also add up to the line sum

| | | |
|---|---|---|
| 3 | 0 | 0 |
| 0 | 1 | 2 |
| 0 | 2 | 1 |

| | | |
|---|---|---|
| 1 | 2 | 0 |
| 0 | 1 | 2 |
| 2 | 0 | 1 |

Traditional magic squares

Definition (Traditional magic square)

A **traditional magic square** is a magic square of order n whose entries are the distinct integers $1, 2, \dots, n^2$

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

The Luo Shu square: the oldest(?) traditional magic square

Counting the traditional magic squares: Out of our scope

The number of traditional magic squares of order n

| n | 1 | 2 | 3 | 4 | 5 | ≥ 6 |
|-----|---|---|---|-----|-----------|----------|
| | 1 | 0 | 1 | 880 | 275305224 | unknown |

Source of integer sequences

- Visit “the On-Line Encyclopedia of Integer Sequences”
<http://www.research.att.com/~njas/sequences/>
- Search with “A006052”

Counting the (semi)magic squares of order 2

Notation

$H_n(t) = \#$ semimagic squares of order n and line sum t

$M_n(t) = \#$ magic squares of order n and line sum t

Example 6.1

$$H_2(t) = t + 1, \quad M_2(t) = \begin{cases} 1 & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd} \end{cases}$$

| | |
|------------------|------------------|
| ♡ | $t - \heartsuit$ |
| $t - \heartsuit$ | ♡ |

| | |
|---------------|---------------|
| $\frac{t}{2}$ | $\frac{t}{2}$ |
| $\frac{t}{2}$ | $\frac{t}{2}$ |

① It's a Kind of Magic

② Semimagic Squares: Integer Points in the Birkhoff–von Neumann Polytope

③ Magic Generating Functions and Constant-Term Identities

④ The Enumeration of Magic Squares

The Birkhoff-von Neumann Polytope

Definition (The n th Birkhoff-von Neumann polytope \mathcal{B}_n)

$$\mathcal{B}_n := \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : \begin{array}{l} x_{jk} \geq 0 \\ \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

- \mathcal{B}_n lives in \mathbb{R}^{n^2}
- The dimension of \mathcal{B}_n is $(n-1)^2$ (Exer 6.3)
- \mathcal{B}_n is the set of all **doubly stochastic matrices** of order n
- The vertices of \mathcal{B}_n are the **permutation matrices** of order n (Exer 6.5)

Example: \mathcal{B}_2

- The definition:

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{l} x_{11}, x_{12}, x_{21}, x_{22} \geq 0 \\ x_{11} + x_{12} = 1, x_{21} + x_{22} = 1 \\ x_{11} + x_{21} = 1, x_{12} + x_{22} = 1 \end{array} \right\}$$

- The element of \mathcal{B}_2 is in the form

$$\begin{pmatrix} \heartsuit & 1 - \heartsuit \\ 1 - \heartsuit & \heartsuit \end{pmatrix} \quad \text{where } 0 \leq \heartsuit \leq 1$$

- This is one-dimensional
- The extreme points are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Birkhoff-von Neumann polytopes and semimagic squares

Definition (The n th Birkhoff-von Neumann polytope \mathcal{B}_n)

$$\mathcal{B}_n := \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : \begin{array}{l} x_{jk} \geq 0 \\ \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

Observation

$$H_n(t) = \#(t\mathcal{B}_n \cap \mathbb{Z}^{n^2}) = L_{\mathcal{B}_n}(t)$$

Theorem 6.2

 $H_n(t)$ is a polynomial in t of degree $(n-1)^2$ Proof: Immediate from Ehrhart's theorem (Thm 3.8) \square Computing $H_n(t)$ by interpolation

Consequence of Thm 6.2

 $H_n(t)$ can be computed by interpolation if we know $(n-1)^2+1$ values $n = 2$: Need to know 2 values

- $H_2(0) = 1$ (Cor 3.15)
- $H_2(1) = 2$ (right?)
- $\therefore H_2(t) = t + 1$

 $n = 3$: Need to know 5 values

- $H_3(0) = 1$ (Cor 3.15)
- Other four values?

Let's look at a general case

The power of reciprocity

Definition ($H_n^\circ(t)$)

$H_n^\circ(t)$ denotes the number of $n \times n$ square matrices with *positive* integer entries summing up to t along each row and column

- $H_n^\circ(t) = H_n(t - n)$ for $t > n$ (Exer 6.6)
- $H_n^\circ(t) = L_{B_n^\circ}(t)$
- $\therefore H_n^\circ(-t) = (-1)^{(n-1)^2} H_n(t)$ (by Reciprocity)
- $H_n^\circ(1) = H_n^\circ(2) = \dots = H_n^\circ(n-1) = 0$ (Exer 6.7)

Thus we obtain the following theorem

Theorem 6.3

The polynomial H_n satisfies $H_n(-n-t) = (-1)^{(n-1)^2} H_n(t)$ and $H_n(-1) = H_n(-2) = \dots = H_n(-n+1) = 0$ \square

Back to $n = 3$

We already know

- $H_3(0) = 1$

From Theorem 6.3

- $H_3(-1) = H_3(-2) = 0$
- $H_3(-3) = (-1)^{(3-1)^2} H_3(0) = 1$

We can also observe

- $H_3(1) = 6$ (by Exer 6.1)

Therefore, by interpolation we get

$$H_3(t) = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$$

The use of Theorem 6.3: The general case

Need to know $(n-1)^2+1$ values of $H_n(t)$

- $H_n(0) = 1$
- $H_n(-1) = H_n(-2) = \dots = H_n(-n+1) = 0$

Still need to know n^2-3n+2 values of $H_n(t)$

- $H_n(-n) = (-1)^{(n-1)^2} H_n(0)$
- Compute $H_n(t)$ for all $t = 1, 2, \dots, (n^2-3n+2)/2$
- $H_n(-n-t) = (-1)^{(n-1)^2} H_n(t)$
for all $t = 1, \dots, (n^2-3n+2)/2-1$

The actual work is to compute $(n^2-3n+2)/2$ values

① It's a Kind of Magic

② Semimagic Squares: Integer Points in the Birkhoff-von Neumann Polytope

③ Magic Generating Functions and Constant-Term Identities

④ The Enumeration of Magic Squares

Compute $H_n(t)$ by means of a generating function

Consider the points in \mathcal{B}_n as column vectors in \mathbb{R}^{n^2} then

$$\mathcal{B}_n = \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^{n^2} : \mathbf{A} \mathbf{x} = \mathbf{b} \right\},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & \cdots & 1 & & & & & \\ & & & 1 & \cdots & 1 & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \cdots & 1 \\ 1 & & & & 1 & & & 1 & \\ & \ddots & & & & \ddots & & & \\ & & 1 & & 1 & & \cdots & \ddots & \\ & & & & & & & & 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

 Compute $H_n(t)$ by means of a generating function (cont'd)

- For a general rational polytope $\mathcal{P} = \{ \mathbf{x} \in \mathbb{R}_{\geq 0}^d : \mathbf{A} \mathbf{x} = \mathbf{b} \}$, we have

$$L_{\mathcal{P}}(t) = \text{const} \left(\frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1})(1 - \mathbf{z}^{\mathbf{c}_2}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d}) \mathbf{z}^{t\mathbf{b}}} \right),$$

where $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_d$ denote the columns of \mathbf{A}

- To keep things as clear as possible, we use z_1, z_2, \dots, z_n for the first n rows of \mathbf{A} (representing the row constraints of \mathcal{B}_n) and w_1, w_2, \dots, w_n for the last n rows of \mathbf{A} (representing the column constraints of \mathcal{B}_n)

 Compute $H_n(t)$ by means of a generating function (further cont'd)

Then, by Theorem 2.13 we get the following

Theorem 6.5

$$H_n(t) = \text{const}_{\substack{z_1, \dots, z_n, \\ w_1, \dots, w_n}} \left(\frac{1}{\prod_{1 \leq j, k \leq n} (1 - z_j w_k) \left(\prod_{1 \leq j \leq n} z_j \prod_{1 \leq k \leq n} w_k \right)^t} \right) \quad \square$$

When $n = 2$, $H_2(t)$ is equal to

$$\text{const}_{\substack{z_1, z_2, \\ w_1, w_2}} \left(\frac{1}{(1 - z_1 w_1)(1 - z_2 w_1)(1 - z_1 w_2)(1 - z_2 w_2) z_1^t z_2^t w_1^t w_2^t} \right)$$

Looking at the generating function more carefully

$$\begin{aligned} H_2(t) &= \text{const}_{\substack{z_1, z_2, \\ w_1, w_2}} \left(\frac{1}{(1 - z_1 w_1)(1 - z_2 w_1)(1 - z_1 w_2)(1 - z_2 w_2) z_1^t z_2^t w_1^t w_2^t} \right) \\ &= \text{const}_{\substack{z_1, z_2, \\ w_1, w_2}} \left(\frac{1}{z_1^t z_2^t} \frac{1}{(1 - z_1 w_1)(1 - z_2 w_1) w_1^t} \frac{1}{(1 - z_1 w_2)(1 - z_2 w_2) w_2^t} \right) \\ &= \text{const}_{z_1, z_2} \left(\frac{1}{z_1^t z_2^t} \text{const}_{w_1} \left(\frac{1}{(1 - z_1 w_1)(1 - z_2 w_1) w_1^t} \right) \right. \\ &\quad \left. \times \text{const}_{w_2} \left(\frac{1}{(1 - z_1 w_2)(1 - z_2 w_2) w_2^t} \right) \right) \\ &= \text{const}_{z_1, z_2} \left(\frac{1}{z_1^t z_2^t} \left(\text{const}_w \frac{1}{(1 - z_1 w)(1 - z_2 w) w^t} \right)^2 \right) \end{aligned}$$

A general consideration

- In general, we get (by Exer 6.8)

$$H_n(t) = \text{const}_{z_1, \dots, z_n} \left(\frac{1}{(z_1 \cdots z_n)^t} \left(\text{const}_w \frac{1}{(1 - z_1 w) \cdots (1 - z_n w) w^t} \right)^n \right)$$

- Let's manipulate the inner expression

$$\text{const}_w \frac{1}{(1 - z_1 w) \cdots (1 - z_n w) w^t}$$

by the partial fraction expansion

A general consideration: Partial fraction expansion

- By partial fraction expansion, we get

$$\frac{1}{(1 - z_1 w) \cdots (1 - z_n w) w^t} = \frac{A_1}{w - \frac{1}{z_1}} + \frac{A_2}{w - \frac{1}{z_2}} + \cdots + \frac{A_n}{w - \frac{1}{z_n}} + \sum_{k=1}^t \frac{B_k}{w^k}$$

- Therefore

$$\text{const}_w \frac{1}{(1 - z_1 w) \cdots (1 - z_n w) w^t} = -A_1 z_1 - A_2 z_2 - \cdots - A_n z_n$$

A general consideration: Partial fraction expansion (cont'd)

- Exercise 6.9 says $A_k = -\frac{z_k^{t+n-2}}{\prod_{j \neq k} (z_k - z_j)}$
- Thus we get the following

Theorem 6.6

$$H_n(t) = \text{const}_{z_1, \dots, z_n} \left(\frac{1}{(z_1 \cdots z_n)^t} \left(\sum_{k=1}^n \frac{z_k^{t+n-1}}{\prod_{j \neq k} (z_k - z_j)} \right)^n \right) \quad \square$$

For $n = 2$

$$\begin{aligned} H_2(t) &= \text{const}_{z_1, z_2} \left(\frac{1}{(z_1 z_2)^t} \left(\frac{z_1^{t+1}}{z_1 - z_2} + \frac{z_2^{t+1}}{z_2 - z_1} \right)^2 \right) \\ &= \text{const}_{z_1, z_2} \left(\frac{z_1^{t+2} z_2^{-t}}{(z_1 - z_2)^2} - 2 \frac{z_1 z_2}{(z_1 - z_2)^2} + \frac{z_1^{-t} z_2^{t+2}}{(z_1 - z_2)^2} \right) \end{aligned}$$

We first compute the constant terms wrt z_1 , then wrt z_2

Preparation for computing the constant terms

- Assume $|z_1| < |z_2|$ (by Exer 6.11)
- Then

$$\frac{1}{z_1 - z_2} = \frac{1}{z_2} \frac{1}{\frac{z_1}{z_2} - 1} = -\frac{1}{z_2} \sum_{k \geq 0} \left(\frac{z_1}{z_2} \right)^k = -\sum_{k \geq 0} \frac{1}{z_2^{k+1}} z_1^k$$

and hence

$$\frac{1}{(z_1 - z_2)^2} = -\frac{d}{dz_1} \left(\frac{1}{z_1 - z_2} \right) = \sum_{k \geq 1} \frac{k}{z_2^{k+1}} z_1^{k-1} = \sum_{k \geq 0} \frac{k+1}{z_2^{k+2}} z_1^k$$

Computing the constant terms (1)

The first term

$$\begin{aligned} \text{const}_{z_1} \left(\frac{z_1^{t+2} z_2^{-t}}{(z_1 - z_2)^2} \right) &= z_2^{-t} \text{const}_{z_1} \left(\frac{z_1^{t+2}}{(z_1 - z_2)^2} \right) \\ &= z_2^{-t} \text{const}_{z_1} \left(z_1^{t+2} \sum_{k \geq 0} \frac{k+1}{z_2^{k+2}} z_1^k \right) \\ &= z_2^{-t} \text{const}_{z_1} \left(\sum_{k \geq 0} \frac{k+1}{z_2^{k+2}} z_1^{k+t+2} \right) = 0 \end{aligned}$$

The second term (Exercise 6.10)

$$\text{const}_{z_1} \left(-2 \frac{z_1 z_2}{(z_1 - z_2)^2} \right) = 0$$

① It's a Kind of Magic

② Semimagic Squares: Integer Points in the Birkhoff–von Neumann Polytope

③ Magic Generating Functions and Constant-Term Identities

④ The Enumeration of Magic Squares

Computing the constant terms (2)

The third term

$$\begin{aligned} \text{const}_{z_1} \left(\frac{z_1^{-t} z_2^{t+2}}{(z_1 - z_2)^2} \right) &= z_2^{t+2} \text{const}_{z_1} \left(z_1^{-t} \sum_{k \geq 0} \frac{k+1}{z_2^{k+2}} z_1^k \right) \\ &= z_2^{t+2} \text{const}_{z_1} \left(\sum_{k \geq 0} \frac{k+1}{z_2^{k+2}} z_1^{k-t} \right) \\ &= z_2^{t+2} \frac{t+1}{z_2^{t+2}} = t+1 \end{aligned}$$

Therefore

$$\begin{aligned} H_2(t) &= \text{const}_{z_1, z_2} \left(\frac{z_1^{t+2} z_2^{-t}}{(z_1 - z_2)^2} - 2 \frac{z_1 z_2}{(z_1 - z_2)^2} + \frac{z_1^{-t} z_2^{t+2}}{(z_1 - z_2)^2} \right) \\ &= \text{const}_{z_2} (0 - 0 + (t+1)) = t+1 \end{aligned}$$

 Quasipolynomiality of $M_n(t)$

- Reminder: $M_n(t) = \#$ magic squares of order n and line sum t
- We have already seen

$$M_2(t) = \begin{cases} 1 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

This is a quasipolynomial

- Similarly to \mathcal{B}_n , we may consider

$$\mathcal{B}'_n := \mathcal{B}_n \cap \left\{ \left(\begin{array}{ccc} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{array} \right) : \begin{array}{l} \sum_j x_{jj} = 1 \\ \sum_j x_{j, n-j+1} = 1 \end{array} \right\}$$

 then $M_n(t) = L_{\mathcal{B}'_n}(t)$

Theorem 6.7 (follows from Ehrhart's theorem)

 $M_n(t)$ is a quasipolynomial in t .


Computing $M_n(t)$ What is $M_3(t)$?

Let's compute with a generating function

| | | |
|-------|-------|-------|
| m_1 | m_2 | m_3 |
| m_4 | m_5 | m_6 |
| m_7 | m_8 | m_9 |

The polytope \mathcal{B}'_3 is determined by

$$m_1 + m_2 + m_3 = t, \quad m_4 + m_5 + m_6 = t, \quad m_7 + m_8 + m_9 = t,$$

 z_1 z_2 z_3

$$m_1 + m_4 + m_7 = t, \quad m_2 + m_5 + m_8 = t, \quad m_3 + m_6 + m_9 = t,$$

 w_1 w_2 w_3

$$m_1 + m_5 + m_9 = t, \quad m_3 + m_5 + m_7 = t$$

 y_1 y_2 For each equality, we use a **variable**Computing $M_n(t)$: Generating functionThen, $M_3(t)$ is the constant term of

$$\frac{1}{(1 - z_1 w_1 y_1)(1 - z_1 w_2)(1 - z_1 w_3 y_2)(1 - z_2 w_1)(1 - z_2 w_2 y_1 y_2)} \\ \times \frac{1}{(1 - z_2 w_3)(1 - z_3 w_1 y_2)(1 - z_3 w_2)(1 - z_3 w_3 y_1)} \\ \times \frac{1}{(z_1 z_2 z_3 w_1 w_2 w_3 y_1 y_2)^t}$$

After some minutes of calculation, we get

$$M_3(t) = \begin{cases} \frac{2}{9} t^2 + \frac{2}{3} t + 1 & \text{if } 3|t, \\ 0 & \text{otherwise} \end{cases}$$