Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(7) Magic squares

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1. It’s a Kind of Magic
2. Semimagic Squares: Integer Points in the Birkhoff–von Neumann Polytope
3. Magic Generating Functions and Constant-Term Identities
4. The Enumeration of Magic Squares

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the denominator of $\mathcal{P}$

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

The goal of this chapter

1. Count the number of magic squares by means of Ehrhart theory

A magic square in an engraving

Melencolia I
Albrecht Dürer (1514)

A magic square in architecture

Temple de la Sagrada Família
Barcelona, Spain

1 It's a Kind of Magic

2 Semimagic Squares: Integer Points in the Birkhoff–von Neumann Polytope

3 Magic Generating Functions and Constant-Term Identities

4 The Enumeration of Magic Squares

You may think these definitions are too weak...

Definition (Semimagic square, Magic square)

- A semimagic square is a square matrix whose entries are nonnegative integers and whose rows and columns sum to the same number (called the line sum)
  - The rows and columns are called lines

3	0	0
0	1	2
0	2	1

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Definition (Semimagic square, Magic square)

- A semimagic square is a square matrix whose entries are nonnegative integers and whose rows and columns sum to the same number (called the line sum)
  - The rows and columns are called lines
- A magic square is a semimagic square whose main diagonals also add up to the line sum

3	0	0
0	1	2
0	2	1

1	2	0
0	1	2
2	0	1

Traditional magic squares

Definition (Traditional magic square)

A traditional magic square is a magic square of order n whose entries are the distinct integers $1, 2, \ldots, n^2$

4	9	2
3	5	7
8	1	6

The Luo Shu square: the oldest(?) traditional magic square

Counting the traditional magic squares: Out of our scope

The number of traditional magic squares of order n

n	1	2	3	4	5	≥ 6
	1	0	1	880	275305224	unknown

Source of integer sequences

- Visit "the On-Line Encyclopedia of Integer Sequences" `http://www.research.att.com/~njas/sequences/`
- Search with "A006052"

Counting the (semi)magic squares of order 2

Notation

$$H_n(t) = \#\text{ semimagic squares of order } n \text{ and line sum } t$$
$$M_n(t) = \#\text{ magic squares of order } n \text{ and line sum } t$$

Example 6.1

Counting the (semi)magic squares of order 2

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Example 6.1

$$H_2(t) = t + 1$$

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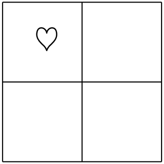

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$\heartsuit$	$t-\heartsuit$
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Example 6.1

$$H_2(t) = t + 1, \quad M_2(t) = \begin{cases} 1 & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd} \end{cases}$$

$\heartsuit$	$t-\heartsuit$
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$\frac{t}{2}$	$\frac{t}{2}$
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The Birkoff–von Neumann Polytope

Definition (The nth Birkoff–von Neumann polytope $\mathcal{B}_n$)

$$\mathcal{B}_n := \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : \begin{array}{l} x_{jk} \geq 0 \\ \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

- $\mathcal{B}_n$ lives in $\mathbb{R}^{n^2}$
- The dimension of $\mathcal{B}_n$ is $(n-1)^2$ (Exer 6.3)
- $\mathcal{B}_n$ is the set of all doubly stochastic matrices of order n
- The vertices of $\mathcal{B}_n$ are the permuatation matrices of order n (Exer 6.5)

Example: $\mathcal{B}_2$

- The definition:

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{l} x_{11}, x_{12}, x_{21}, x_{22} \geq 0 \\ x_{11} + x_{12} = 1, x_{21} + x_{22} = 1 \\ x_{11} + x_{21} = 1, x_{12} + x_{22} = 1 \end{array} \right\}$$

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- The element of $\mathcal{B}_2$ is in the form

$$\begin{pmatrix} \heartsuit & 1 - \heartsuit \\ 1 - \heartsuit & \heartsuit \end{pmatrix} \qquad \text{where } 0 \leq \heartsuit \leq 1$$

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- The element of $\mathcal{B}_2$ is in the form

$$\begin{pmatrix} \heartsuit & 1 - \heartsuit \\ 1 - \heartsuit & \heartsuit \end{pmatrix} \qquad \text{where } 0 \leq \heartsuit \leq 1$$

- This is one-dimensional
- The extreme points are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Birkoff–von Neumann polytopes and semimagic squares

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Observation

$$H_n(t) = \# \left(t\mathcal{B}_n \cap \mathbb{Z}^{n^2} \right) = L_{\mathcal{B}_n}(t)$$

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Observation

$$H_n(t) = \#\left(t\mathcal{B}_n \cap \mathbb{Z}^{n^2}\right) = L_{\mathcal{B}_n}(t)$$

Theorem 6.2

$H_n(t)$ is a polynomial in t of degree $(n-1)^2$

Proof: Immediate from Ehrhart's theorem (Thm 3.8) □

Computing $H_n(t)$ by interpolation

Consequence of Thm 6.2

$H_n(t)$ can be computed by interpolation if we know $(n-1)^2+1$ values

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- $H_2(1) = 2$ (right?)

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- $\therefore H_2(t) = t + 1$

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$n = 3$: Need to know 5 values

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Computing $H_n(t)$ by interpolation

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$H_n(t)$ can be computed by interpolation if we know $(n-1)^2+1$ values

$n = 2$: Need to know 2 values

- $H_2(0) = 1$ (Cor 3.15)
- $H_2(1) = 2$ (right?)
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$n = 3$: Need to know 5 values

- $H_3(0) = 1$ (Cor 3.15)
- Other four values?

Let's look at a general case

The power of reciprocity

Definition ($H_n^\circ(t)$)

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- $H_n^\circ(1) = H_n^\circ(2) = \cdots = H_n^\circ(n-1) = 0$ (Exer 6.7)

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- $H_n^\circ(1) = H_n^\circ(2) = \cdots = H_n^\circ(n-1) = 0$ (Exer 6.7)

Thus we obtain the following theorem

Theorem 6.3

The polynomial H_n satisfies $H_n(-n-t) = (-1)^{(n-1)^2} H_n(t)$ and $H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0$ □

Back to $n = 3$

We already know

- $H_3(0) = 1$

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From Theorem 6.3

- $H_3(-1) = H_3(-2) = 0$
- $H_3(-3) = (-1)^{(3-1)^2} H_3(0) = 1$

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We can also observe

- $H_3(1) = 6$ (by Exer 6.1)

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- $H_3(1) = 6$ (by Exer 6.1)

Therefore, by interpolation we get

$$H_3(t) = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$$

The use of Theorem 6.3: The general case

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- $H_n(0) = 1$
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Still need to know n^2-3n+2 values of $H_n(t)$

The use of Theorem 6.3: The general case

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Still need to know n^2-3n+2 values of $H_n(t)$

- $H_n(-n) = (-1)^{(n-1)^2} H_n(0)$
- Compute $H_n(t)$ for all $t = 1, 2, \dots, (n^2-3n+2)/2$

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- $H_n(-n-t) = (-1)^{(n-1)^2} H_n(t)$
  for all $t = 1, \ldots, (n^2-3n+2)/2-1$

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The actual work is to compute $(n^2-3n+2)/2$ values

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Compute $H_n(t)$ by means of a generating function

Consider the points in $\mathcal{B}_n$ as column vectors in $\mathbb{R}^{n^2}$ then

$$\mathcal{B}_n = \left\{ \mathbf{x} \in \mathbb{R}^{n^2}_{\geq 0} : \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & \cdots & 1 & & & & & & & \\ & & & 1 & \cdots & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & \cdots & 1 \\ 1 & & & 1 & & & & 1 & & \\ & \ddots & & & \ddots & & \cdots & & \ddots & \\ & & 1 & & & 1 & & & & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Compute $H_n(t)$ by means of a generating function (cont'd)

- For a general rational polytope $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$, we have

$$L_{\mathcal{P}}(t) = \text{const} \left(\frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1})(1 - \mathbf{z}^{\mathbf{c}_2}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{t\mathbf{b}}} \right),$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_d$ denote the columns of $\mathbf{A}$

Compute $H_n(t)$ by means of a generating function (cont'd)

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$$L_{\mathcal{P}}(t) = \text{const}\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{t\mathbf{b}}} \right),$$

  where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_d$ denote the columns of $\mathbf{A}$
- To keep things as clear as possible,
  we use $z_1, z_2, \ldots, z_n$ for the first n rows of $\mathbf{A}$
  (representing the row constraints of $\mathcal{B}_n$) and
  $w_1, w_2, \ldots, w_n$ for the last n rows of $\mathbf{A}$
  (representing the column constraints of $\mathcal{B}_n$)

Compute $H_n(t)$ by means of a generating function (further cont'd)

Then, by Theorem 2.13 we get the following

Theorem 6.5

$$H_n(t) = \operatorname*{const}_{\substack{z_1,\dots,z_n,\\ w_1,\dots,w_n}} \left(\frac{1}{\prod_{1\le j,k\le n} (1 - z_j w_k) \left(\prod_{1\le j\le n} z_j \prod_{1\le k\le n} w_k \right)^t} \right) \qquad \square$$

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When $n = 2$, $H_2(t)$ is equal to

$$\operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{(1-z_1w_1)(1-z_2w_1)(1-z_1w_2)(1-z_2w_2)\, z_1^t z_2^t w_1^t w_2^t} \right)$$

Looking at the generating function more carefully

$H_2(t)$

$$= \operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{(1-z_1w_1)(1-z_2w_1)(1-z_1w_2)(1-z_2w_2)\, z_1^t z_2^t w_1^t w_2^t} \right)$$

Looking at the generating function more carefully

$$
\begin{aligned}
&H_2(t)\\
&= \operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{(1-z_1w_1)(1-z_2w_1)(1-z_1w_2)(1-z_2w_2)\, z_1^t z_2^t w_1^t w_2^t} \right)\\
&= \operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{z_1^t z_2^t} \frac{1}{(1-z_1w_1)(1-z_2w_1)\, w_1^t} \frac{1}{(1-z_1w_2)(1-z_2w_2)\, w_2^t} \right)
\end{aligned}
$$

Looking at the generating function more carefully

$$
\begin{aligned}
&H_2(t)\\
&= \operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{(1-z_1w_1)(1-z_2w_1)(1-z_1w_2)(1-z_2w_2)\, z_1^t z_2^t w_1^t w_2^t} \right)\\
&= \operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{z_1^t z_2^t} \frac{1}{(1-z_1w_1)(1-z_2w_1)\, w_1^t} \frac{1}{(1-z_1w_2)(1-z_2w_2)\, w_2^t} \right)\\
&= \operatorname*{const}_{z_1,z_2} \left(\frac{1}{z_1^t z_2^t} \operatorname*{const}_{w_1} \left(\frac{1}{(1-z_1w_1)(1-z_2w_1)\, w_1^t} \right) \right.\\
&\qquad\qquad \left. \times \operatorname*{const}_{w_2} \left(\frac{1}{(1-z_1w_2)(1-z_2w_2)\, w_2^t} \right) \right)
\end{aligned}
$$

Looking at the generating function more carefully

$$
\begin{aligned}
&H_2(t) \\
&= \operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{(1-z_1w_1)(1-z_2w_1)(1-z_1w_2)(1-z_2w_2)\, z_1^t z_2^t w_1^t w_2^t} \right) \\
&= \operatorname*{const}_{\substack{z_1,z_2,\\ w_1,w_2}} \left(\frac{1}{z_1^t z_2^t} \frac{1}{(1-z_1w_1)(1-z_2w_1)\, w_1^t} \frac{1}{(1-z_1w_2)(1-z_2w_2)\, w_2^t} \right) \\
&= \operatorname*{const}_{z_1,z_2} \left(\frac{1}{z_1^t z_2^t} \operatorname*{const}_{w_1} \left(\frac{1}{(1-z_1w_1)(1-z_2w_1)\, w_1^t} \right) \right. \\
&\qquad\qquad \left. \times \operatorname*{const}_{w_2} \left(\frac{1}{(1-z_1w_2)(1-z_2w_2)\, w_2^t} \right) \right) \\
&= \operatorname*{const}_{z_1,z_2} \left(\frac{1}{z_1^t z_2^t} \left(\operatorname*{const}_{w} \frac{1}{(1-z_1w)(1-z_2w)\, w^t} \right)^2 \right)
\end{aligned}
$$

A general consideration

- In general, we get (by Exer 6.8)

$$H_n(t) = \operatorname*{const}_{z_1,\ldots,z_n} \left(\frac{1}{(z_1 \cdots z_n)^t} \left(\operatorname*{const}_{w} \frac{1}{(1 - z_1 w) \cdots (1 - z_n w) w^t} \right)^n \right)$$

- Let's manipulate the inner expression

$$\operatorname*{const}_{w} \frac{1}{(1 - z_1 w) \cdots (1 - z_n w) w^t}$$

by the partial fraction expansion

A general consideration: Partial fraction expansion

- By partial fraction expansion, we get

$$\frac{1}{(1-z_1w)\cdots(1-z_nw)w^t} = \frac{A_1}{w-\frac{1}{z_1}} + \frac{A_2}{w-\frac{1}{z_2}} + \cdots + \frac{A_n}{w-\frac{1}{z_n}} + \sum_{k=1}^{t}\frac{B_k}{w^k}$$

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- Therefore

$$\operatorname*{const}_{w} \frac{1}{(1-z_1w)\cdots(1-z_nw)w^t} = -A_1z_1 - A_2z_2 - \cdots - A_nz_n$$

A general consideration: Partial fraction expansion (cont'd)

- Exercise 6.9 says $A_k = -\dfrac{z_k^{t+n-2}}{\prod_{j \neq k}(z_k - z_j)}$

A general consideration: Partial fraction expansion (cont'd)

- Exercise 6.9 says $A_k = -\frac{z_k^{t+n-2}}{\prod_{j\neq k}(z_k - z_j)}$
- Thus we get the following

Theorem 6.6

$$H_n(t) = \operatorname*{const}_{z_1,\dots,z_n} \left(\frac{1}{(z_1 \cdots z_n)^t} \left(\sum_{k=1}^{n} \frac{z_k^{t+n-1}}{\prod_{j\neq k}(z_k - z_j)} \right)^n \right)$$

□

A general consideration: Partial fraction expansion (cont'd)

- Exercise 6.9 says $A_k = -\frac{z_k^{t+n-2}}{\prod_{j \neq k}(z_k - z_j)}$
- Thus we get the following

Theorem 6.6

$$H_n(t) = \underset{z_1,\ldots,z_n}{\text{const}} \left(\frac{1}{(z_1 \cdots z_n)^t} \left(\sum_{k=1}^{n} \frac{z_k^{t+n-1}}{\prod_{j \neq k}(z_k - z_j)} \right)^n \right) \qquad \square$$

For $n = 2$

$$H_2(t) = \underset{z_1,z_2}{\text{const}} \left(\frac{1}{(z_1 z_2)^t} \left(\frac{z_1^{t+1}}{z_1 - z_2} + \frac{z_2^{t+1}}{z_2 - z_1} \right)^2 \right)$$

A general consideration: Partial fraction expansion (cont'd)

- Exercise 6.9 says $A_k = -\dfrac{z_k^{t+n-2}}{\prod_{j\neq k}(z_k - z_j)}$
- Thus we get the following

Theorem 6.6

$$H_n(t) = \operatorname*{const}_{z_1,\ldots,z_n} \left(\frac{1}{(z_1 \cdots z_n)^t} \left(\sum_{k=1}^{n} \frac{z_k^{t+n-1}}{\prod_{j\neq k}(z_k - z_j)} \right)^n \right)$$ □

For $n = 2$

$$H_2(t) = \operatorname*{const}_{z_1,z_2} \left(\frac{1}{(z_1 z_2)^t} \left(\frac{z_1^{t+1}}{z_1 - z_2} + \frac{z_2^{t+1}}{z_2 - z_1} \right)^2 \right)$$

$$= \operatorname*{const}_{z_1,z_2} \left(\frac{z_1^{t+2} z_2^{-t}}{(z_1 - z_2)^2} - 2\frac{z_1 z_2}{(z_1 - z_2)^2} + \frac{z_1^{-t} z_2^{t+2}}{(z_1 - z_2)^2} \right)$$

A general consideration: Partial fraction expansion (cont'd)

- Exercise 6.9 says $A_k = -\frac{z_k^{t+n-2}}{\prod_{j\neq k}(z_k - z_j)}$
- Thus we get the following

Theorem 6.6

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For $n = 2$

$$H_2(t) = \operatorname*{const}_{z_1,z_2} \left(\frac{1}{(z_1 z_2)^t} \left(\frac{z_1^{t+1}}{z_1 - z_2} + \frac{z_2^{t+1}}{z_2 - z_1} \right)^2 \right)$$

$$= \operatorname*{const}_{z_1,z_2} \left(\frac{z_1^{t+2} z_2^{-t}}{(z_1 - z_2)^2} - 2\frac{z_1 z_2}{(z_1 - z_2)^2} + \frac{z_1^{-t} z_2^{t+2}}{(z_1 - z_2)^2} \right)$$

We first compute the constant terms wrt z_1, then wrt z_2

Preparation for computing the constant terms

- Assume $|z_1| < |z_2|$ (by Exer 6.11)

Preparation for computing the constant terms

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- Then

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Preparation for computing the constant terms

- Assume $|z_1| < |z_2|$ (by Exer 6.11)
- Then

$$\frac{1}{z_1 - z_2} = \frac{1}{z_2}\frac{1}{\frac{z_1}{z_2} - 1} = -\frac{1}{z_2}\sum_{k\geq 0}\left(\frac{z_1}{z_2}\right)^k$$

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Preparation for computing the constant terms

- Assume $|z_1| < |z_2|$ (by Exer 6.11)
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$$\frac{1}{z_1 - z_2} = \frac{1}{z_2}\frac{1}{\frac{z_1}{z_2} - 1} = -\frac{1}{z_2}\sum_{k\geq 0}\left(\frac{z_1}{z_2}\right)^k = -\sum_{k\geq 0}\frac{1}{z_2^{k+1}} z_1^k$$

and hence

$$\frac{1}{(z_1 - z_2)^2} = -\frac{d}{dz_1}\left(\frac{1}{z_1 - z_2}\right)$$

Preparation for computing the constant terms

- Assume $|z_1| < |z_2|$ (by Exer 6.11)
- Then

$$\frac{1}{z_1 - z_2} = \frac{1}{z_2}\frac{1}{\frac{z_1}{z_2} - 1} = -\frac{1}{z_2}\sum_{k\geq 0}\left(\frac{z_1}{z_2}\right)^k = -\sum_{k\geq 0}\frac{1}{z_2^{k+1}}\, z_1^k$$

and hence

$$\frac{1}{(z_1 - z_2)^2} = -\frac{d}{dz_1}\left(\frac{1}{z_1 - z_2}\right) = \sum_{k\geq 1}\frac{k}{z_2^{k+1}}\, z_1^{k-1}$$

Preparation for computing the constant terms

- Assume $|z_1| < |z_2|$ (by Exer 6.11)
- Then

$$\frac{1}{z_1 - z_2} = \frac{1}{z_2}\frac{1}{\frac{z_1}{z_2} - 1} = -\frac{1}{z_2}\sum_{k\geq 0}\left(\frac{z_1}{z_2}\right)^k = -\sum_{k\geq 0}\frac{1}{z_2^{k+1}}\, z_1^k$$

and hence

$$\frac{1}{(z_1 - z_2)^2} = -\frac{d}{dz_1}\left(\frac{1}{z_1 - z_2}\right) = \sum_{k\geq 1}\frac{k}{z_2^{k+1}}\, z_1^{k-1} = \sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\, z_1^k$$

Computing the constant terms (1)

The first term

$$\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2} z_2^{-t}}{(z_1 - z_2)^2}\right)$$

Computing the constant terms (1)

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Computing the constant terms (1)

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$$\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}z_2^{-t}}{(z_1-z_2)^2}\right) = z_2^{-t}\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}}{(z_1-z_2)^2}\right)$$
$$= z_2^{-t}\operatorname*{const}_{z_1}\left(z_1^{t+2}\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^k\right)$$

Computing the constant terms (1)

The first term

$$\begin{aligned}\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}z_2^{-t}}{(z_1-z_2)^2}\right) &= z_2^{-t}\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}}{(z_1-z_2)^2}\right)\\ &= z_2^{-t}\operatorname*{const}_{z_1}\left(z_1^{t+2}\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k}\right)\\ &= z_2^{-t}\operatorname*{const}_{z_1}\left(\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k+t+2}\right)\end{aligned}$$

Computing the constant terms (1)

The first term

$$\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}z_2^{-t}}{(z_1-z_2)^2}\right) = z_2^{-t}\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}}{(z_1-z_2)^2}\right)$$

$$= z_2^{-t}\operatorname*{const}_{z_1}\left(z_1^{t+2}\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k}\right)$$

$$= z_2^{-t}\operatorname*{const}_{z_1}\left(\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k+t+2}\right) = 0$$

Computing the constant terms (1)

The first term

$$\begin{aligned}\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}z_2^{-t}}{(z_1-z_2)^2}\right) &= z_2^{-t}\operatorname*{const}_{z_1}\left(\frac{z_1^{t+2}}{(z_1-z_2)^2}\right)\\ &= z_2^{-t}\operatorname*{const}_{z_1}\left(z_1^{t+2}\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k}\right)\\ &= z_2^{-t}\operatorname*{const}_{z_1}\left(\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k+t+2}\right) = 0\end{aligned}$$

The second term (Exercise 6.10)

$$\operatorname*{const}_{z_1}\left(-2\frac{z_1 z_2}{(z_1-z_2)^2}\right) = 0$$

Computing the constant terms (2)

The third term

$$\operatorname*{const}_{z_1}\left(\frac{z_1^{-t} z_2^{t+2}}{(z_1 - z_2)^2}\right)$$

Computing the constant terms (2)

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$$\operatorname*{const}_{z_1}\left(\frac{z_1^{-t}z_2^{t+2}}{(z_1-z_2)^2}\right) = z_2^{t+2}\operatorname*{const}_{z_1}\left(z_1^{-t}\sum_{k\ge 0}\frac{k+1}{z_2^{k+2}}\,z_1^k\right)$$

Computing the constant terms (2)

The third term

$$\operatorname*{const}_{z_1}\left(\frac{z_1^{-t}z_2^{t+2}}{(z_1-z_2)^2}\right) = z_2^{t+2}\operatorname*{const}_{z_1}\left(z_1^{-t}\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k}\right)$$

$$= z_2^{t+2}\operatorname*{const}_{z_1}\left(\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k-t}\right)$$

Computing the constant terms (2)

The third term

$$\operatorname*{const}_{z_1}\left(\frac{z_1^{-t}z_2^{t+2}}{(z_1-z_2)^2}\right) = z_2^{t+2}\operatorname*{const}_{z_1}\left(z_1^{-t}\sum_{k\ge 0}\frac{k+1}{z_2^{k+2}}\,z_1^k\right)$$

$$= z_2^{t+2}\operatorname*{const}_{z_1}\left(\sum_{k\ge 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k-t}\right)$$

$$= z_2^{t+2}\,\frac{t+1}{z_2^{t+2}}$$

Computing the constant terms (2)

The third term

$$\begin{aligned}
\operatorname*{const}_{z_1}\left(\frac{z_1^{-t} z_2^{t+2}}{(z_1 - z_2)^2}\right) &= z_2^{t+2} \operatorname*{const}_{z_1}\left(z_1^{-t} \sum_{k \geq 0} \frac{k+1}{z_2^{k+2}} z_1^k\right) \\
&= z_2^{t+2} \operatorname*{const}_{z_1}\left(\sum_{k \geq 0} \frac{k+1}{z_2^{k+2}} z_1^{k-t}\right) \\
&= z_2^{t+2} \frac{t+1}{z_2^{t+2}} = t+1
\end{aligned}$$

Computing the constant terms (2)

The third term

$$\operatorname*{const}_{z_1}\left(\frac{z_1^{-t}z_2^{t+2}}{(z_1-z_2)^2}\right) = z_2^{t+2}\operatorname*{const}_{z_1}\left(z_1^{-t}\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k}\right)$$

$$= z_2^{t+2}\operatorname*{const}_{z_1}\left(\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k-t}\right)$$

$$= z_2^{t+2}\,\frac{t+1}{z_2^{t+2}} = t+1$$

Therefore

$$H_2(t) = \operatorname*{const}_{z_1,z_2}\left(\frac{z_1^{t+2}z_2^{-t}}{(z_1-z_2)^2} - 2\frac{z_1z_2}{(z_1-z_2)^2} + \frac{z_1^{-t}z_2^{t+2}}{(z_1-z_2)^2}\right)$$

Computing the constant terms (2)

The third term

$$\operatorname*{const}_{z_1}\left(\frac{z_1^{-t}z_2^{t+2}}{(z_1-z_2)^2}\right) = z_2^{t+2}\operatorname*{const}_{z_1}\left(z_1^{-t}\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^k\right)$$
$$= z_2^{t+2}\operatorname*{const}_{z_1}\left(\sum_{k\geq 0}\frac{k+1}{z_2^{k+2}}\,z_1^{k-t}\right)$$
$$= z_2^{t+2}\,\frac{t+1}{z_2^{t+2}} = t+1$$

Therefore

$$H_2(t) = \operatorname*{const}_{z_1,z_2}\left(\frac{z_1^{t+2}z_2^{-t}}{(z_1-z_2)^2} - 2\frac{z_1z_2}{(z_1-z_2)^2} + \frac{z_1^{-t}z_2^{t+2}}{(z_1-z_2)^2}\right)$$
$$= \operatorname*{const}_{z_2}\left(0-0+(t+1)\right) = t+1$$

1. It's a Kind of Magic
2. Semimagic Squares: Integer Points in the Birkhoff–von Neumann Polytope
3. Magic Generating Functions and Constant-Term Identities
4. The Enumeration of Magic Squares

Quasipolynomiality of $M_n(t)$

- Reminder: $M_n(t) = \#$ magic squares of order n and line sum t
- We have already seen

$$M_2(t) = \begin{cases} 1 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

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  This is a quasipolynomial
- Similarly to $\mathcal{B}_n$, we may consider

$$\mathcal{B}'_n := \mathcal{B}_n \cap \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} : \begin{array}{l} \sum_j x_{jj} = 1 \\ \sum_j x_{j,n-j+1} = 1 \end{array} \right\}$$

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  then $M_n(t) = L_{\mathcal{B}'_n}(t)$

Quasipolynomiality of $M_n(t)$

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- Similarly to $\mathcal{B}_n$, we may consider

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  then $M_n(t) = L_{\mathcal{B}'_n}(t)$

Theorem 6.7 (follows from Ehrhart's theorem)

$M_n(t)$ is a quasipolynomial in t. □

Computing $M_n(t)$

m_1	m_2	m_3
m_4	m_5	m_6
m_7	m_8	m_9

What is $M_3(t)$?

Let's compute with a generating function

The polytope $\mathcal{B}'_3$ is determined by

$$m_1 + m_2 + m_3 = t\,, \quad m_4 + m_5 + m_6 = t\,, \quad m_7 + m_8 + m_9 = t\,,$$

$$m_1 + m_4 + m_7 = t\,, \quad m_2 + m_5 + m_8 = t\,, \quad m_3 + m_6 + m_9 = t\,,$$

$$m_1 + m_5 + m_9 = t\,, \quad m_3 + m_5 + m_7 = t$$

Computing $M_n(t)$

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z_1 z_2 z_3

$$m_1 + m_4 + m_7 = t\,,\quad m_2 + m_5 + m_8 = t\,,\quad m_3 + m_6 + m_9 = t\,,$$

w_1 w_2 w_3

$$m_1 + m_5 + m_9 = t\,,\quad m_3 + m_5 + m_7 = t$$

y_1 y_2

For each equality, we use a variable

Computing $M_n(t)$: Generating function

Then, $M_3(t)$ is the constant term of

$$\frac{1}{(1-z_1w_1y_1)(1-z_1w_2)(1-z_1w_3y_2)(1-z_2w_1)(1-z_2w_2y_1y_2)}$$
$$\times\frac{1}{(1-z_2w_3)(1-z_3w_1y_2)(1-z_3w_2)(1-z_3w_3y_1)}$$
$$\times\frac{1}{(z_1z_2z_3w_1w_2w_3y_1y_2)^t}$$

Computing $M_n(t)$: Generating function

Then, $M_3(t)$ is the constant term of

$$\frac{1}{(1-z_1w_1y_1)(1-z_1w_2)(1-z_1w_3y_2)(1-z_2w_1)(1-z_2w_2y_1y_2)}$$
$$\times \frac{1}{(1-z_2w_3)(1-z_3w_1y_2)(1-z_3w_2)(1-z_3w_3y_1)}$$
$$\times \frac{1}{(z_1z_2z_3w_1w_2w_3y_1y_2)^t}$$

After some minutes of calculation, we get

$$M_3(t) = \begin{cases} \frac{2}{9}t^2 + \frac{2}{3}t + 1 & \text{if } 3|t, \\ 0 & \text{otherwise} \end{cases}$$