Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(6) Face Numbers and the Dehn–Sommerville Relations in Ehrhartian Terms

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Important theorems from the previous lectures

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

The goal of this chapter

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1. To prove Dehn–Sommerville relations, a set of fascinating identities, which give linear relations among the face numbers f_k
2. To unify the Dehn–Sommerville relations with Ehrhart–Macdonald reciprocity

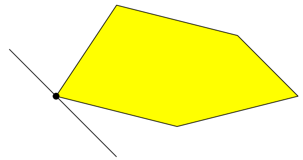
Faces of a convex polytope: recap

$\mathcal{P} \subseteq \mathbb{R}^d$ a convex polytope

Definition (Face)

$\mathcal{F}$ is a face of $\mathcal{P}$ if $\exists$ a valid inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ for $\mathcal{P}$ s.t.

$$\mathcal{F} = \mathcal{P} \cap \{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = b\}$$



Remark

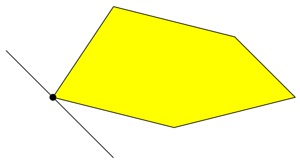
- Every face of a convex polytope is also a convex polytope
- $\mathcal{P}$ and $\varnothing$ are faces of $\mathcal{P}$

Face numbers

$\mathcal{P}$ a d-polytope, fixed

Definition (Face number)

$f_k :=$ the number of k-dimensional faces of $\mathcal{P}$, $k = 0, 1, \ldots, d$

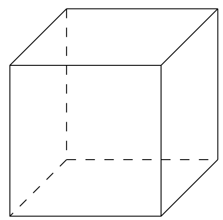


$f_0 = 5, f_1 = 5, f_2 = 1$

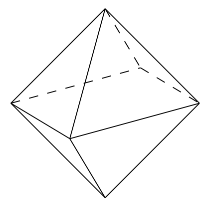
Simple polytopes

Definition (Simple polytope)

The d-polytope $\mathcal{P}$ is simple if each vertex of $\mathcal{P}$ lies on precisely d edges of $\mathcal{P}$



simple



non-simple

Dehn–Sommerville relations

Fundamental linear relations among face numbers

Theorem 5.1 (Dehn–Sommerville relations: Dehn '05, Sommerville '27)

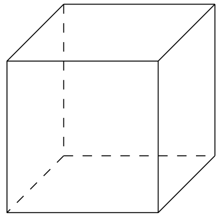
For a simple d-polytope $\mathcal{P}$ and $0 \le k \le d$,

$$f_k = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} f_j$$

Remarks

- This holds for *all* simple polytopes
- Doesn't hold for non-simple polytopes in general (Exer 5.11)
- We will prove for rational polytopes by means of Ehrhart theory

Dehn–Sommerville relations: Example



$f_0 = 8$
$f_1 = 12$
$f_2 = 6$
$f_3 = 1$

$$f_0 = \binom{3}{3} f_0 = f_0$$

$$f_1 = \binom{3}{2} f_0 - \binom{2}{2} f_1 = 3f_0 - f_1$$

$$\Rightarrow 3f_0 = 2f_1$$

$$f_2 = \binom{3}{1} f_0 - \binom{2}{1} f_1 + \binom{1}{1} f_2$$

$$= 3f_0 - 2f_1 + f_2 \Rightarrow 3f_0 = 2f_1$$

$$f_3 = \binom{3}{0} f_0 - \binom{2}{0} f_1 + \binom{1}{0} f_2 - \binom{0}{0} f_3$$

$$= f_0 - f_1 + f_2 - f_3$$

$$\Rightarrow f_0 - f_1 + f_2 = 2f_3 = 2$$

The Euler relation

Theorem 5.2 (The Euler relation)

$\mathcal{P}$ a convex d-polytope $\Rightarrow$

$$\sum_{j=0}^{d} (-1)^j f_j = 1$$

Proof for simple polytopes via Theorem 5.1:

- Theorem 5.1 for $k = d$ gives

$$1 = f_d = \sum_{j=0}^{d} (-1)^j \binom{d-j}{d-d} f_j = \sum_{j=0}^{d} (-1)^j f_j \qquad \square$$

The Euler relation (cont'd)

Proof for rational polytopes via Ehrhart–Mcdonald's reciprocity:

- By Ehrhart–Mcdonald's reciprocity

$$L_{\mathcal{P}}(t) = \sum_{\mathcal{F} \subseteq \mathcal{P}} L_{\mathcal{F}^\circ}(t) = \sum_{\mathcal{F} \subseteq \mathcal{P}} (-1)^{\dim \mathcal{F}} L_{\mathcal{F}}(-t)$$

  where the sums are over all *nonempty* faces
- $\mathrm{const}(L_{\mathcal{F}}(t)) = 1$ for all $\mathcal{F}$ (Exer 3.27)
- The constant term gives

$$1 = \mathrm{const}(L_{\mathcal{P}}(t)) = \sum_{\mathcal{F} \subseteq \mathcal{P}} (-1)^{\dim \mathcal{F}} \mathrm{const}(L_{\mathcal{F}}(-t))$$

$$= \sum_{\mathcal{F} \subseteq \mathcal{P}} (-1)^{\dim \mathcal{F}} 1 = \sum_{j=0}^{d} (-1)^j f_j \qquad \square$$

Toward a generalization

$\mathcal{P}$ a convex polytope

Definition

$$F_k(t) := \sum_{\substack{\mathcal{F} \subseteq \mathcal{P} \\ \dim \mathcal{F} = k}} L_{\mathcal{F}}(t)$$

Remark: Suppose $\mathcal{P}$ is rational

- $F_k(t)$ is a quasipolynomial
- $F_k(0) = f_k$ $\quad (\because L_{\mathcal{F}}(0) = 1)$
- The leading coefficient of $F_k(t)$ is the relative volume of the k-skeleton of $\mathcal{P}$
  - The k-skeleton of $\mathcal{P}$ is the union of all k-faces of $\mathcal{P}$

Today's main theorem

Theorem 5.3 (McMullen '77)

$\mathcal{P}$ a simple rational d-polytope and $0 \le k \le d \Rightarrow$

$$F_k(t) = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} F_j(-t)$$

Proof of Theorem 5.1 via Theorem 5.3:

- Looking at the constant terms, we obtain

$$f_k = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} f_j \qquad \square$$

"Proof" of Ehrhart–Macdonald's reciprocity via Theorem 5.3

for *simple* polytopes

$$L_{\mathcal{P}}(t) = F_d(t) = \sum_{j=0}^{d} (-1)^j \binom{d-j}{d-d} F_j(-t) = (-1)^d \sum_{j=0}^{d} (-1)^{d-j} F_j(-t)$$

$$\begin{aligned} \therefore L_{\mathcal{P}}(-t) &= (-1)^d \sum_{j=0}^{d} (-1)^{d-j} F_j(t) \\ &= (-1)^d \sum_{j=0}^{d} (-1)^{d-j} \sum_{\substack{\mathcal{F} \subseteq \mathcal{P} \\ \dim \mathcal{F} = j}} L_{\mathcal{F}}(t) \\ &= (-1)^d \sum_{\mathcal{F} \subseteq \mathcal{P}} (-1)^{d - \dim \mathcal{F}} L_{\mathcal{F}}(t) = (-1)^d L_{\mathcal{P}^\circ}(t) \end{aligned}$$

by the Inclusion–Exclusion (á la Exer 5.4)

Proof of Theorem 5.3

$\mathcal{P}$ a simple d-polytope, $\mathcal{F}$ a k-face of $\mathcal{P}$

- Since $\forall\ \mathbf{m} \in \mathcal{F} \cap \mathbb{Z}^d\ \exists!\ \mathcal{G} \subseteq \mathcal{F}$: $\mathbf{m} \in \mathcal{G}^\circ$,

$$L_{\mathcal{F}}(t) = \sum_{\mathcal{G}\subseteq\mathcal{F}} L_{\mathcal{G}^\circ}(t)$$

- By Ehrhart–Macdonald's reciprocity

$$L_{\mathcal{F}}(t) = \sum_{\mathcal{G}\subseteq\mathcal{F}} (-1)^{\dim\mathcal{G}} L_{\mathcal{G}}(-t) = \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathcal{G}\subseteq\mathcal{F}\\ \dim\mathcal{G}=j}} L_{\mathcal{G}}(-t)$$

Now we take the sum over all k-faces

Proof of Theorem 5.3, cont'd

$$F_k(t) = \sum_{\substack{\mathcal{F}\subseteq\mathcal{P}\\ \dim\mathcal{F}=k}} \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathcal{G}\subseteq\mathcal{F}\\ \dim\mathcal{G}=j}} L_{\mathcal{G}}(-t) = \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathcal{F}\subseteq\mathcal{P}\\ \dim\mathcal{F}=k}} \sum_{\substack{\mathcal{G}\subseteq\mathcal{F}\\ \dim\mathcal{G}=j}} L_{\mathcal{G}}(-t)$$

$$= \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathcal{G}\subseteq\mathcal{P}\\ \dim\mathcal{G}=j}} f_k(\mathcal{P}/\mathcal{G}) L_{\mathcal{G}}(-t)$$

$$= \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathcal{G}\subseteq\mathcal{P}\\ \dim\mathcal{G}=j}} \binom{d-j}{d-k} L_{\mathcal{G}}(-t) \qquad \text{(Exer 5.4)}$$

$$= \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} F_j(-t)$$

where $f_k(\mathcal{P}/\mathcal{G})$ is the number of k-faces of $\mathcal{P}$ containing $\mathcal{G}$ □

A consequence of Theorem 5.3 (?)

- From Theorem 5.3

$$L_{\mathcal{P}}(t) = F_d(t) = \sum_{j=0}^{d} (-1)^j F_j(-t)$$

- Indeed, this holds for any rational polytope (even non-simple)

$$L_{\mathcal{P}}(t) = \sum_{\mathcal{F}\subseteq\mathcal{P}} L_{\mathcal{F}^\circ}(t)$$

$$= \sum_{\mathcal{F}\subseteq\mathcal{P}} (-1)^{\dim\mathcal{F}} L_{\mathcal{F}}(-t) \qquad \text{(reciprocity)}$$

$$= \sum_{j=0}^{d} (-1)^j \sum_{\substack{\mathcal{F}\subseteq\mathcal{P}\\ \dim\mathcal{F}=j}} L_{\mathcal{F}}(-t) = \sum_{j=0}^{d} (-1)^j F_j(-t)$$

Counting the integer points on the boundary

- By Ehrhart–Macdonald's reciprocity,

$$(-1)^d F_d(-t) = (-1)^d L_{\mathcal{P}}(-t) = L_{\mathcal{P}^\circ}(t)$$

- Therefore

$$L_{\mathcal{P}}(t) - L_{\mathcal{P}^\circ}(t) = \left(\sum_{j=0}^{d}(-1)^j F_j(-t)\right) - (-1)^d F_d(-t)$$
$$= \sum_{j=0}^{d-1}(-1)^j F_j(-t)$$

- The LHS counts the number of integer points on the boundary of $t\,\mathcal{P}$

Sum of every second terms of the Ehrhart polynomial

- Let $L_{\mathcal{P}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_0$ ($\mathcal{P}$ integral)
- Then $L_{\mathcal{P}^\circ}(t) = c_d\, t^d - c_{d-1}\, t^{d-1} + \cdots + (-1)^d c_0$
- Hence

$$L_{\mathcal{P}}(t) - L_{\mathcal{P}^\circ}(t) = 2c_{d-1}\, t^{d-1} + 2c_{d-3}\, t^{d-3} + \cdots$$

where this sum ends with $2c_0$ if d is odd and $2c_1 t$ if d is even

Theorem 5.4

$$c_{d-1}\, t^{d-1} + c_{d-3}\, t^{d-3} + \cdots = \frac{1}{2}\sum_{j=0}^{d-1}(-1)^j F_j(-t) \qquad \square$$

We know $c_d = \operatorname{vol}\mathcal{P}$. How about other coefficients?

- Let $F_j(t) = c_{j,j}\, t^j + c_{j,j-1} t^{j-1} + \cdots + c_{j,0}$

Corollary 5.5

k and d are of different parity $\Rightarrow c_k = \frac{1}{2}\sum_{j=0}^{d-1}(-1)^{j+k} c_{j,k}$

Example

- $c_{d-1} = \frac{1}{2} \sum_{\mathcal{F} \text{ a facet of } \mathcal{P}}$ the leading coeff's of $L_{\mathcal{F}}(t)$

Return to continuous volumes

Lemma 3.19 (recap)

$S \subset \mathbb{R}^d$ d-dimensional $\Rightarrow$

$$\operatorname{vol} S = \lim_{t \to \infty} \frac{1}{t^d} \cdot \#\left(tS \cap \mathbb{Z}^d\right)$$

One issue

S is not d-dimensional $\Rightarrow$ $\operatorname{vol} S = 0$ by definition

Motivation

We still would like to compute the volume of smaller-dimensional objects, in the relative sense

Relative volume

Setup

- $S \subset \mathbb{R}^d$ of dimension $m < d$
- $\operatorname{span} S = \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in S,\ \lambda \in \mathbb{R}\}$, the affine span of S
- Consider the sublattice $(\operatorname{span} S) \cap \mathbb{Z}^d$
- The relative volume of S is the volume relative to $(\operatorname{span} S) \cap \mathbb{Z}^d$

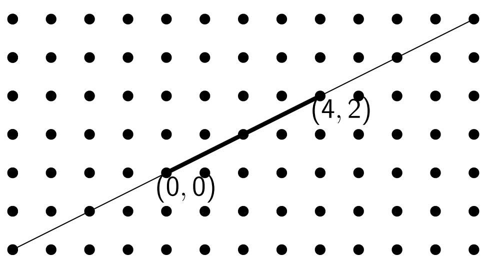
Definition or Proposition (Relative volume)

The relative volume of S is

$$\operatorname{vol} S = \lim_{t \to \infty} \frac{1}{t^m} \cdot \#\left(tS \cap \mathbb{Z}^d\right)$$

Convention: $\operatorname{vol} S$ represents the relative volume of S, not the volume of S when $m < d$

Example

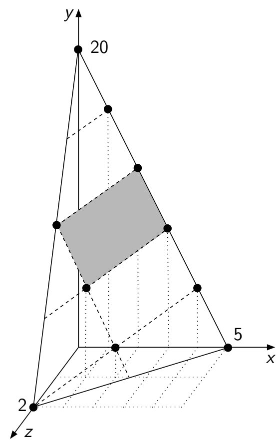


- $\operatorname{span} L = \{(x, y) \in \mathbb{R}^2 : y = x/2\}$
- $\operatorname{vol} L = 2$

Cf.

- the Euclidean length of $L = 2\sqrt{5}$

Example 2



- $\mathcal{P}$ = the triangle defined by $\frac{x}{5} + \frac{y}{20} + \frac{z}{2} = 1$, $x \geq 0, y \geq 0, z \geq 0$
- $\operatorname{vol} \mathcal{P} = 5$

Cf.

- the Euclidean area of $\mathcal{P}$ $= 15\sqrt{13}$
- the Euclidean area of the shaded region $= 3\sqrt{13}$

Relation to the coefficients of Ehrhart polynomials

- Let $L_{\mathcal{P}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_0$ ($\mathcal{P}$ integral)
- We saw $c_{d-1} = \dfrac{1}{2} \displaystyle\sum_{\mathcal{F} \text{ a facet of } \mathcal{P}}$ the leading coeff of $L_{\mathcal{F}}(t)$
- We know the leading coeff of $L_{\mathcal{F}}(t)$ is $\operatorname{vol} \mathcal{F}$

Therefore

Theorem 5.6

$$c_{d-1} = \frac{1}{2} \sum_{\mathcal{F} \text{ a facet of} \mathcal{P}} \operatorname{vol} \mathcal{F}$$

□