Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(5) Reciprocity

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May 21, 2009

"Last updated: 2009/05/20 9:58"

We saw several examples...

For several integral d-polytopes $\mathcal{P}$ we saw

$$L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t)$$

This holds in general, also for rational polytopes

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

We're going to prove this theorem today

Reciprocity in combinatorics

Theorem 4.1 belongs to a class of famous reciprocity theorems

A common theme in combinatorics

Begin with an interesting object P, and

1. define a counting function $f(t)$ attached to P that makes physical sense for positive integer values of t;
2. recognize the function f as a polynomial in t;
3. substitute negative integral values of t into the counting function f, and recognize $f(-t)$ as a counting function of a new object Q

In this course

P a polytope; Q its interior

1. Generating Functions for Somewhat Irrational Cones
2. Stanley's Reciprocity Theorem for Rational Cones
3. Ehrhart–Macdonald Reciprocity for Rational Polytopes
4. The Ehrhart Series of Reflexive Polytopes

Integer-point transforms of a somewhat irrational pointed cone

Theorem 4.2

- $\mathcal{K}$ the simplicial cone generated by $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d \in \mathbb{Z}^d$
- $\mathbf{v} \in \mathbb{R}^d$ s.t. the boundary of $\mathbf{v} + \mathcal{K}$ contains no integer point

$$\Rightarrow \quad \sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \, \sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z})$$

- Reminder: $\sigma_S(\mathbf{z}) = \displaystyle\sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$
- Notation: $\dfrac{1}{\mathbf{z}} = \left(\dfrac{1}{z_1}, \dfrac{1}{z_2}, \dots, \dfrac{1}{z_d}\right)$ when $\mathbf{z} = (z_1, z_2, \dots, z_d)$

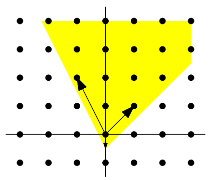
Theorem 4.2: Example

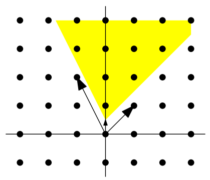
$\mathbf{w}_1 = (1,1)$, $\mathbf{w}_2 = (-1,2)$, $\mathbf{v} = (0,-1/2)$

By Corollary 3.6

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} = \frac{1+z_2+z_2^2}{(1-z_1z_2)(1-z_1^{-1}z_2^2)}$$

$$\sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = \frac{1+z_2^{-1}+z_2^{-2}}{(1-z_1^{-1}z_2^{-1})(1-z_1z_2^{-2})}$$

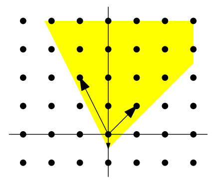


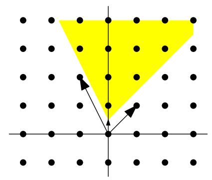


Theorem 4.2: Example (continued)

$\mathbf{w}_1 = (1,1)$, $\mathbf{w}_2 = (-1,2)$, $\mathbf{v} = (0,-1/2)$

$$\begin{aligned}\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) &= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\ &= \frac{z_2+z_2^2+z_2^3}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\ &= \frac{z_2+z_2^2+z_2^3}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\ &\quad \times \frac{z_2^{-3}}{(z_1^{-1}z_2^{-1})(z_1z_2^{-2})} \\ &= \frac{z_2^{-2}+z_2^{-1}+1}{(z_1^{-1}z_2^{-1}-1)(z_1z_2^{-2}-1)} \\ &= \sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right)\end{aligned}$$





Proof of Thm 4.2 (1)

- By Corollary 3.6

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})},$$

where

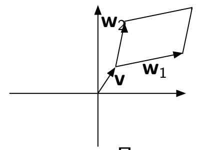
$$\Pi = \{\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_d\mathbf{w}_d : 0 < \lambda_1, \lambda_2, \ldots, \lambda_d < 1\}$$

- Similarly, $-\mathbf{v}+\mathcal{K}$ satisfies the assumption of Corollary 3.6 and hence

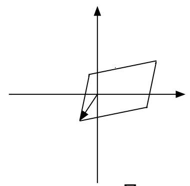
$$\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})}$$

Proof of Thm 4.2 (2)

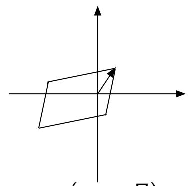
Exercise 4.2: $\mathbf{v}+\Pi = -(-\mathbf{v}+\Pi) + \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_d$



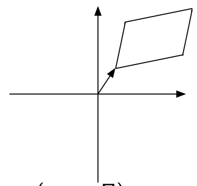
$\mathbf{v}+\Pi$



$-\mathbf{v}+\Pi$



$-(-\mathbf{v}+\Pi)$



$-(-\mathbf{v}+\Pi)+\mathbf{w}_1+\mathbf{w}_2$

Proof of Thm 4.2 (3)

Therefore, by Exer 3.6

$$\begin{aligned}
\sigma_{\mathbf{v}+\Pi}(\mathbf{z}) &= \sigma_{-(-\mathbf{v}+\Pi)+\mathbf{w}_1+\mathbf{w}_2+\cdots+\mathbf{w}_d}(\mathbf{z}) \\
&= \sigma_{-(-\mathbf{v}+\Pi)}(\mathbf{z})\,\mathbf{z}^{\mathbf{w}_1}\mathbf{z}^{\mathbf{w}_2}\cdots\mathbf{z}^{\mathbf{w}_d} \\
&= \sigma_{-\mathbf{v}+\Pi}\left(\tfrac{1}{\mathbf{z}}\right)\mathbf{z}^{\mathbf{w}_1}\mathbf{z}^{\mathbf{w}_2}\cdots\mathbf{z}^{\mathbf{w}_d} \\
\therefore\ \sigma_{\mathbf{v}+\Pi}\left(\frac{1}{\mathbf{z}}\right) &= \sigma_{-\mathbf{v}+\Pi}(\mathbf{z})\,\mathbf{z}^{-\mathbf{w}_1}\mathbf{z}^{-\mathbf{w}_2}\cdots\mathbf{z}^{-\mathbf{w}_d}
\end{aligned}$$

Proof of Thm 4.2 (4)

Therefore,

$$\begin{aligned}
\sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) &= \frac{\sigma_{\mathbf{v}+\Pi}\left(\frac{1}{\mathbf{z}}\right)}{(1-\mathbf{z}^{-\mathbf{w}_1})(1-\mathbf{z}^{-\mathbf{w}_2})\cdots(1-\mathbf{z}^{-\mathbf{w}_d})} \\
&= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})\,\mathbf{z}^{-\mathbf{w}_1}\mathbf{z}^{-\mathbf{w}_2}\cdots\mathbf{z}^{-\mathbf{w}_d}}{(1-\mathbf{z}^{-\mathbf{w}_1})(1-\mathbf{z}^{-\mathbf{w}_2})\cdots(1-\mathbf{z}^{-\mathbf{w}_d})} \\
&= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(\mathbf{z}^{\mathbf{w}_1}-1)(\mathbf{z}^{\mathbf{w}_2}-1)\cdots(\mathbf{z}^{\mathbf{w}_d}-1)} \\
&= (-1)^d\frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})} \\
&= (-1)^d\,\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) \qquad \square
\end{aligned}$$

Stanley's reciprocity theorem

Theorem 4.3 (Stanley reciprocity)

$\mathcal{K}$ a rational d-cone with the origin as apex $\Rightarrow$

$$\sigma_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \, \sigma_{\mathcal{K}^\circ}(\mathbf{z})$$

Proof of Theorem 4.3

- Triangulate $\mathcal{K}$ into the simplicial cones $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$
- From Exer 3.14, $\exists\, \mathbf{v} \in \mathbb{R}^d$ s.t.
  - $\mathcal{K}^\circ \cap \mathbb{Z}^d = (\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$
  - $\partial(\mathbf{v} + \mathcal{K}_j) \cap \mathbb{Z}^d = \varnothing$ for all $j = 1, \ldots, m$
  - $\partial(-\mathbf{v} + \mathcal{K}_j) \cap \mathbb{Z}^d = \varnothing$ for all $j = 1, \ldots, m$
- Then $\mathcal{K} \cap \mathbb{Z}^d = (-\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$ (Exer 4.3)
- Then by Theorem 4.1

$$\sigma_{\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sigma_{-\mathbf{v}+\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sum_{j=1}^{m} \sigma_{-\mathbf{v}+\mathcal{K}_j}\left(\tfrac{1}{\mathbf{z}}\right) = \sum_{j=1}^{m} (-1)^d \sigma_{\mathbf{v}+\mathcal{K}_j}(\mathbf{z})$$
$$= (-1)^d \sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = (-1)^d \sigma_{\mathcal{K}^\circ}(\mathbf{z}) \qquad \square$$

Definition (Ehrhart series of the interior of a rational polytope)

The Ehrhart series of the interior of a rational polytope $\mathcal{P}$ is

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) := \sum_{t\geq 1} L_{\mathcal{P}^\circ}(t)\, z^t$$

Theorem 4.4

$\mathcal{P}$ a convex rational d-polytope $\Rightarrow$

$$\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}^\circ}(z)$$

Proof of Theorem 4.4

- By Lemma 3.10

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{t\geq 0} L_{\mathcal{P}}(t)\, z^t = \sigma_{\mathrm{cone}(\mathcal{P})}(1,1,\dots,1,z)$$

- Similarly

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = \sum_{t\geq 1} L_{\mathcal{P}^\circ}(t)\, z^t = \sigma_{(\mathrm{cone}(\mathcal{P}))^\circ}(1,1,\dots,1,z)$$

- By applying Stanley's reciprocity (Thm 4.3) to cone($\mathcal{P}$) we obtain

$$\sigma_{(\mathrm{cone}(\mathcal{P}))^\circ}(1,1,\dots,1,z) = (-1)^{d+1}\,\sigma_{\mathrm{cone}(\mathcal{P})}\left(1,1,\dots,1,\frac{1}{z}\right)$$

$\square$

Proof of Ehrhart–Macdonald's reciprocity (Thm 4.1)

- By Exer 4.6

$$\begin{aligned} -\sum_{t\geq 1} L_{\mathcal{P}}(-t)\, z^t &= \sum_{t\leq 0} L_{\mathcal{P}}(-t)\, z^t \\ &= \sum_{t\geq 0} L_{\mathcal{P}}(t)\, z^{-t} = \mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) \end{aligned}$$

- Then by Theorem 4.4

$$\sum_{t\geq 1} L_{\mathcal{P}^\circ}(t)\, z^t = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^{d} \sum_{t\geq 1} L_{\mathcal{P}}(-t)\, z^t$$

- Comparing the coefficients, we obtain the theorem $\square$

Degression: the degree of an integral polytope

Definition (Degree of an integral polytope)

For an integral d-polytope $\mathcal{P}$ with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}},$$

the degree of $\mathcal{P}$ is the largest k s.t. $h_k \neq 0$

Theorem 4.5

The degree of an integral d-polytope $\mathcal{P}$ is $k \Leftrightarrow$ $(d-k+1)\mathcal{P}$ is the smallest integer dilate of $\mathcal{P}$ that contains an interior lattice point

Proof of Theorem 4.5

- We use Theorem 3.18
- The degree of $\mathcal{P}$ is $k \Leftrightarrow$
  $L_\mathcal{P}(-1) = L_\mathcal{P}(-2) = \cdots = L_\mathcal{P}(-(d-k)) = 0$ and
  $L_\mathcal{P}(-(d-k+1)) \neq 0$
- By the Ehrhart–Macdonald reciprocity, this is equivalent to $\mathcal{P}^\circ, (2\mathcal{P})^\circ, \ldots, ((d-k)\mathcal{P})^\circ$ contains no lattice point and $((d-k+1)\mathcal{P})^\circ$ contains a lattice point $\square$

Reflexive polytopes

Definition (Reflexive polytope)

A polytope $\mathcal{P}$ that contains the origin in its interior is reflexive if it is integral and has the hyperplane description

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \leq \mathbf{1}\right\},$$

where $\mathbf{A}$ is an integral matrix

Example: d-Crosspolytopes

Palindromy of the Ehrhart series of a reflexive polytope

Theorem 4.6 (Hibi's palindromic theorem)

$\mathcal{P}$ an integral d-polytope that contains the origin in its interior and that has the Ehrhart series

$$\mathrm{Ehr}_\mathcal{P}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}}$$

$\Rightarrow$ $\mathcal{P}$ reflexive if and only if $h_k = h_{d-k}\ \forall\, 0 \leq k \leq \frac{d}{2}$

Example:

$$\mathrm{Ehr}_\diamond(z) = \frac{\sum_{k=0}^d \binom{d}{k} z^k}{(1-z)^{d+1}}, \quad \binom{d}{k} = \binom{d}{d-k}$$

A lemma we use for the proof of Thm 4.6

Lemma 4.7

$a_1, a_2, \ldots, a_d, b \in \mathbb{Z}$ satisfy $\gcd(a_1, a_2, \ldots, a_d, b) = 1$ and $b > 1$
$\Rightarrow \exists\, c, t \in \mathbb{Z}_{>0}$ s.t.

- $tb < c < (t+1)b$,
- $\{(m_1, \ldots, m_d) \in \mathbb{Z}^d : a_1 m_1 + \cdots + a_d m_d = c\} \neq \varnothing$

Proof:

- Let $g = \gcd(a_1, a_2, \ldots, a_d)$
- $\exists\, k \in \mathbb{Z}$ and $t \in \mathbb{Z}_{>0}$ s.t. $kg - tb = 1$ $\quad (\because \gcd(g, b) = 1)$
- Let $c = kg$
- Then $tb < c < (t+1)b$ $\quad (\because kg - tb = 1, b > 1)$
- Since $g = \gcd(a_1, a_2, \ldots, a_d)$, $\exists\, m_1, m_2, \ldots, m_d \in \mathbb{Z}$ s.t.

$$a_1 m_1 + a_2 m_2 + \cdots + a_d m_d = kg = c \qquad \square$$

Proof of Thm 4.6

Claim

$\mathcal{P}$ reflexive $\Leftrightarrow$

- $\mathcal{P}^\circ \cap \mathbb{Z}^d = \{\mathbf{0}\}$
- $(t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d = t\mathcal{P} \cap \mathbb{Z}^d$ for all $t \in \mathbb{Z}_{>0}$

Proof of $\Rightarrow$: Exercise 4.12
Proof of $\Leftarrow$: Assume $\mathcal{P}$ satisfies the conditions on the RHS

- Let $H = \{\mathbf{x} \in \mathbb{R}^d : a_1 x_1 + a_2 x_2 + \cdots + a_d x_d = b\}$ define a facet of $\mathcal{P}$ (wlog $\gcd(a_1, a_2, \ldots, a_d, b) = 1$)
- $\exists$ no lattice point between tH and $(t+1)H$ $\quad$ (by Exer 4.13)
- $\therefore \{\mathbf{x} \in \mathbb{Z}^d : tb < a_1 x_1 + a_2 x_2 + \cdots + a_d x_d < (t+1)b\} = \varnothing$
- $\therefore b = 1$ $\quad$ (by Lem 4.7)

$\square$

Proof of Thm 4.6 (cont'd)

- By Theorem 4.4

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = \frac{h_0 z^{d+1} + h_1 z^d + \cdots + h_{d-1} z^2 + h_d z}{(1-z)^{d+1}}$$

- By Claim, $\mathcal{P}$ reflexive iff $\mathrm{Ehr}_{\mathcal{P}^\circ}(z)$ is equal to

$$\sum_{t \geq 1} L_{\mathcal{P}}(t-1)\, z^t = z \sum_{t \geq 0} L_{\mathcal{P}}(t)\, z^t = z\,\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d z^{d+1} + h_{d-1} z^d + \cdots + h_1 z^2 + h_0 z}{(1-z)^{d+1}}$$

$\square$

Summary

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

Theorem 4.6 (Hibi's palindromic theorem)

$\mathcal{P}$ an integral d-polytope that contains the origin in its interior and that has the Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d z^d + h_{d-1} z^{d-1} + \cdots + h_1 z + h_0}{(1-z)^{d+1}}$$

$\Rightarrow \mathcal{P}$ reflexive if and only if $h_k = h_{d-k}\ \forall\, 0 \leq k \leq \frac{d}{2}$

Rest of the course

- Dehn–Sommerville relations
  - Relations among the face numbers of a polytope
  - Ehrhart–Macdonald's reciprocity will be used as a tool
- Magic squares
  - Concrete example of a lattice point counting
  - Ehrhart–Macdonald's reciprocity will be used as a tool
- Finite Fourier series
  - Study of periodic functions
- Fourier–Dedekind sums
  - Appeared in Ehrhart quasipolynomials
  - Computational aspects
- Decomposition of a polytope into cones (Brion's theorem)
  - A magical relation between a polytope and its vertex cones
  - Computational aspects