Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(5) Reciprocity

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1. Generating Functions for Somewhat Irrational Cones

2. Stanley's Reciprocity Theorem for Rational Cones

3. Ehrhart–Macdonald Reciprocity for Rational Polytopes

4. The Ehrhart Series of Reflexive Polytopes

We saw several examples...

For several integral d-polytopes $\mathcal{P}$ we saw

$$L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t)$$

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$$L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t)$$

This holds in general, also for rational polytopes

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

We're going to prove this theorem today

Theorem 4.1 belongs to a class of famous reciprocity theorems

A common theme in combinatorics

Begin with an interesting object P, and

1. define a counting function $f(t)$ attached to P that makes physical sense for positive integer values of t;
2. recognize the function f as a polynomial in t;
3. substitute negative integral values of t into the counting function f, and recognize $f(-t)$ as a counting function of a new object Q

In this course

P a polytope; Q its interior

Integer-point transforms of a somewhat irrational pointed cone

Theorem 4.2

- $\mathcal{K}$ the simplicial cone generated by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$
- $\mathbf{v} \in \mathbb{R}^d$ s.t. the boundary of $\mathbf{v} + \mathcal{K}$ contains no integer point

$$\Rightarrow \quad \sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \, \sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z})$$

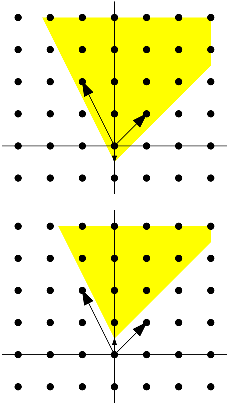
- Reminder: $\sigma_S(\mathbf{z}) = \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$
- Notation: $\frac{1}{\mathbf{z}} = \left(\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_d}\right)$ when $\mathbf{z} = (z_1, z_2, \ldots, z_d)$

Theorem 4.2: Example

$\mathbf{w}_1 = (1, 1)$, $\mathbf{w}_2 = (-1, 2)$, $\mathbf{v} = (0, -1/2)$

By Corollary 3.6

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1 - z_1 z_2)(1 - z_1^{-1} z_2^2)}$$



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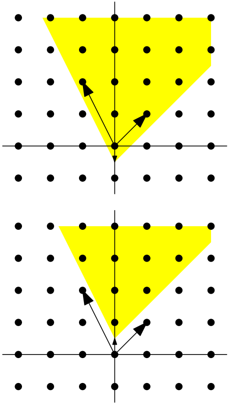

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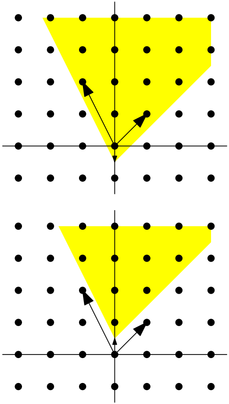
$$= \frac{1+z_2+z_2^2}{(1-z_1z_2)(1-z_1^{-1}z_2^2)}$$

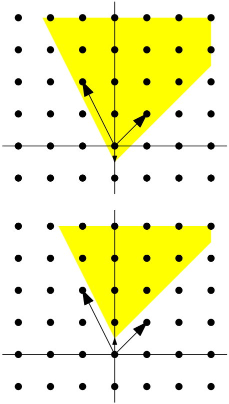
$$\sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = \frac{1+z_2^{-1}+z_2^{-2}}{(1-z_1^{-1}z_2^{-1})(1-z_1z_2^{-2})}$$



Theorem 4.2: Example (continued)

$\mathbf{w}_1 = (1, 1)$, $\mathbf{w}_2 = (-1, 2)$, $\mathbf{v} = (0, -1/2)$

$$\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(1 - z_1 z_2)(1 - z_1^{-1} z_2^2)}$$



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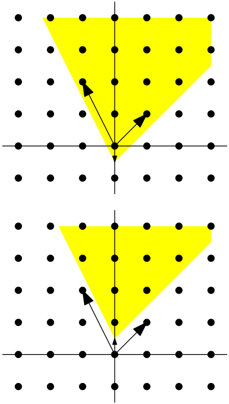

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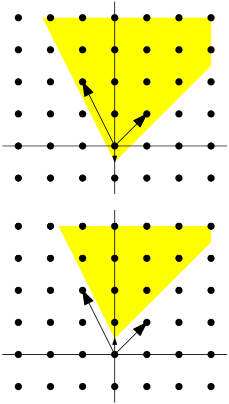
$$\begin{aligned}\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) &= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\ &= \frac{z_2+z_2^2+z_2^3}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\ &= \frac{z_2+z_2^2+z_2^3}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\ &\quad \times \frac{z_2^{-3}}{(z_1^{-1}z_2^{-1})(z_1z_2^{-2})}\end{aligned}$$



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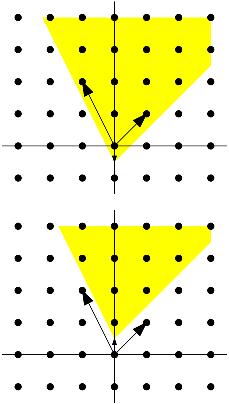
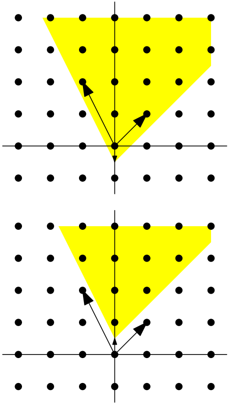


Theorem 4.2: Example (continued)

$\mathbf{w}_1 = (1,1)$, $\mathbf{w}_2 = (-1,2)$, $\mathbf{v} = (0,-1/2)$

$$\begin{aligned}
\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) &= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\
&= \frac{z_2+z_2^2+z_2^3}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\
&= \frac{z_2+z_2^2+z_2^3}{(1-z_1z_2)(1-z_1^{-1}z_2^2)} \\
&\quad \times \frac{z_2^{-3}}{(z_1^{-1}z_2^{-1})(z_1z_2^{-2})} \\
&= \frac{z_2^{-2}+z_2^{-1}+1}{(z_1^{-1}z_2^{-1}-1)(z_1z_2^{-2}-1)} \\
&= \sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right)
\end{aligned}$$



Proof of Thm 4.2 (1)

- By Corollary 3.6

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})},$$

where

$$\Pi = \{\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_d\mathbf{w}_d : 0 < \lambda_1, \lambda_2, \ldots, \lambda_d < 1\}$$

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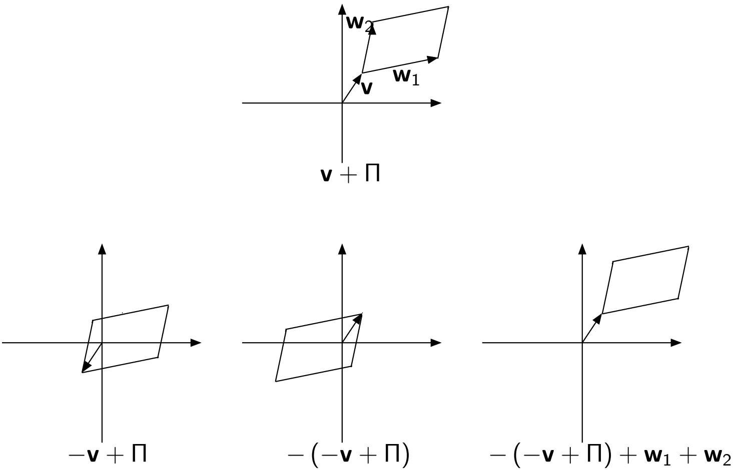
- Similarly, $-\mathbf{v} + \mathcal{K}$ satisfies the assumption of Corollary 3.6 and hence

$$\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})}$$

Proof of Thm 4.2 (2)

Exercise 4.2: $\mathbf{v} + \Pi = -(-\mathbf{v} + \Pi) + \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_d$



Proof of Thm 4.2 (3)

Therefore,

$$\sigma_{\mathbf{v}+\Pi}(\mathbf{z}) = \sigma_{-(-\mathbf{v}+\Pi)+\mathbf{w}_1+\mathbf{w}_2+\cdots+\mathbf{w}_d}(\mathbf{z})$$

Proof of Thm 4.2 (3)

Therefore,

$$\begin{aligned}\sigma_{\mathbf{v}+\Pi}(\mathbf{z}) &= \sigma_{-(-\mathbf{v}+\Pi)+\mathbf{w}_1+\mathbf{w}_2+\cdots+\mathbf{w}_d}(\mathbf{z}) \\ &= \sigma_{-(-\mathbf{v}+\Pi)}(\mathbf{z})\,\mathbf{z}^{\mathbf{w}_1}\mathbf{z}^{\mathbf{w}_2}\cdots\mathbf{z}^{\mathbf{w}_d}\end{aligned}$$

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Therefore, by Exer 3.6

$$\begin{aligned}
\sigma_{\mathbf{v}+\Pi}(\mathbf{z}) &= \sigma_{-(-\mathbf{v}+\Pi)+\mathbf{w}_1+\mathbf{w}_2+\cdots+\mathbf{w}_d}(\mathbf{z}) \\
&= \sigma_{-(-\mathbf{v}+\Pi)}(\mathbf{z})\, \mathbf{z}^{\mathbf{w}_1}\mathbf{z}^{\mathbf{w}_2} \cdots \mathbf{z}^{\mathbf{w}_d} \\
&= \sigma_{-\mathbf{v}+\Pi}\left(\frac{1}{\mathbf{z}}\right) \mathbf{z}^{\mathbf{w}_1}\mathbf{z}^{\mathbf{w}_2} \cdots \mathbf{z}^{\mathbf{w}_d}
\end{aligned}$$

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Proof of Thm 4.2 (4)

Therefore,

$$\sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = \frac{\sigma_{\mathbf{v}+\Pi}\left(\frac{1}{\mathbf{z}}\right)}{\left(1-\mathbf{z}^{-\mathbf{w}_1}\right)\left(1-\mathbf{z}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{-\mathbf{w}_d}\right)}$$

□

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$$= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})\,\mathbf{z}^{-\mathbf{w}_1}\mathbf{z}^{-\mathbf{w}_2}\cdots\mathbf{z}^{-\mathbf{w}_d}}{\left(1-\mathbf{z}^{-\mathbf{w}_1}\right)\left(1-\mathbf{z}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{-\mathbf{w}_d}\right)}$$

□

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$$= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})\,\mathbf{z}^{-\mathbf{w}_1}\mathbf{z}^{-\mathbf{w}_2}\cdots\mathbf{z}^{-\mathbf{w}_d}}{\left(1-\mathbf{z}^{-\mathbf{w}_1}\right)\left(1-\mathbf{z}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{-\mathbf{w}_d}\right)}$$

$$= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{\left(\mathbf{z}^{\mathbf{w}_1}-1\right)\left(\mathbf{z}^{\mathbf{w}_2}-1\right)\cdots\left(\mathbf{z}^{\mathbf{w}_d}-1\right)}$$

□

Proof of Thm 4.2 (4)

Therefore,

$$\begin{aligned}\sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) &= \frac{\sigma_{\mathbf{v}+\Pi}\left(\frac{1}{\mathbf{z}}\right)}{\left(1-\mathbf{z}^{-\mathbf{w}_1}\right)\left(1-\mathbf{z}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{-\mathbf{w}_d}\right)} \\ &= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})\,\mathbf{z}^{-\mathbf{w}_1}\mathbf{z}^{-\mathbf{w}_2}\cdots\mathbf{z}^{-\mathbf{w}_d}}{\left(1-\mathbf{z}^{-\mathbf{w}_1}\right)\left(1-\mathbf{z}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{-\mathbf{w}_d}\right)} \\ &= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{\left(\mathbf{z}^{\mathbf{w}_1}-1\right)\left(\mathbf{z}^{\mathbf{w}_2}-1\right)\cdots\left(\mathbf{z}^{\mathbf{w}_d}-1\right)} \\ &= (-1)^d\frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{\left(1-\mathbf{z}^{\mathbf{w}_1}\right)\left(1-\mathbf{z}^{\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{w}_d}\right)}\end{aligned}$$

□

Proof of Thm 4.2 (4)

Therefore,

$$\begin{aligned}
\sigma_{\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) &= \frac{\sigma_{\mathbf{v}+\Pi}\left(\frac{1}{\mathbf{z}}\right)}{\left(1-\mathbf{z}^{-\mathbf{w}_1}\right)\left(1-\mathbf{z}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{-\mathbf{w}_d}\right)} \\
&= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})\,\mathbf{z}^{-\mathbf{w}_1}\mathbf{z}^{-\mathbf{w}_2}\cdots\mathbf{z}^{-\mathbf{w}_d}}{\left(1-\mathbf{z}^{-\mathbf{w}_1}\right)\left(1-\mathbf{z}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{-\mathbf{w}_d}\right)} \\
&= \frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{\left(\mathbf{z}^{\mathbf{w}_1}-1\right)\left(\mathbf{z}^{\mathbf{w}_2}-1\right)\cdots\left(\mathbf{z}^{\mathbf{w}_d}-1\right)} \\
&= (-1)^d\,\frac{\sigma_{-\mathbf{v}+\Pi}(\mathbf{z})}{\left(1-\mathbf{z}^{\mathbf{w}_1}\right)\left(1-\mathbf{z}^{\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{w}_d}\right)} \\
&= (-1)^d\,\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z})
\end{aligned}$$

□

Stanley's reciprocity theorem

Theorem 4.3 (Stanley reciprocity)

$\mathcal{K}$ a rational d-cone with the origin as apex $\Rightarrow$

$$\sigma_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \, \sigma_{\mathcal{K}^\circ}(\mathbf{z})$$

Proof of Theorem 4.3

- Triangulate $\mathcal{K}$ into the simplicial cones $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$

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- From Exer 3.14, $\exists\ \mathbf{v} \in \mathbb{R}^d$ s.t.
  - $\mathcal{K}^\circ \cap \mathbb{Z}^d = (\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$
  - $\partial\,(\mathbf{v} + \mathcal{K}_j) \cap \mathbb{Z}^d = \varnothing$ for all $j = 1, \ldots, m$
  - $\partial\,(-\mathbf{v} + \mathcal{K}_j) \cap \mathbb{Z}^d = \varnothing$ for all $j = 1, \ldots, m$

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- Then $\mathcal{K} \cap \mathbb{Z}^d = (-\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$ (Exer 4.3)

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- Then $\mathcal{K} \cap \mathbb{Z}^d = (-\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$ (Exer 4.3)
- Then

$$\sigma_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = \sigma_{-\mathbf{v}+\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right)$$

□

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- Then $\mathcal{K} \cap \mathbb{Z}^d = (-\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$ (Exer 4.3)
- Then

$$\sigma_{\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sigma_{-\mathbf{v}+\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sum_{j=1}^{m} \sigma_{-\mathbf{v}+\mathcal{K}_j}\left(\tfrac{1}{\mathbf{z}}\right)$$

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- Then by Theorem 4.1

$$\sigma_{\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sigma_{-\mathbf{v}+\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sum_{j=1}^{m} \sigma_{-\mathbf{v}+\mathcal{K}_j}\left(\tfrac{1}{\mathbf{z}}\right) = \sum_{j=1}^{m} (-1)^d \sigma_{\mathbf{v}+\mathcal{K}_j}(\mathbf{z})$$

□

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$$= (-1)^d\, \sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z})$$

□

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- Then $\mathcal{K} \cap \mathbb{Z}^d = (-\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$ (Exer 4.3)
- Then by Theorem 4.1

$$\sigma_{\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sigma_{-\mathbf{v}+\mathcal{K}}\left(\tfrac{1}{\mathbf{z}}\right) = \sum_{j=1}^{m} \sigma_{-\mathbf{v}+\mathcal{K}_j}\left(\tfrac{1}{\mathbf{z}}\right) = \sum_{j=1}^{m} (-1)^d \sigma_{\mathbf{v}+\mathcal{K}_j}(\mathbf{z})$$

$$= (-1)^d \sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = (-1)^d \sigma_{\mathcal{K}^\circ}(\mathbf{z}) \qquad \square$$

Definition (Ehrhart series of the interior of a rational polytope)

The Ehrhart series of the interior of a rational polytope $\mathcal{P}$ is

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) := \sum_{t \geq 1} L_{\mathcal{P}^\circ}(t)\, z^t$$

Theorem 4.4

$\mathcal{P}$ a convex rational d-polytope $\Rightarrow$

$$\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^{d+1}\, \mathrm{Ehr}_{\mathcal{P}^\circ}(z)$$

Proof of Theorem 4.4

- By Lemma 3.10

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{t \geq 0} L_{\mathcal{P}}(t)\, z^t = \sigma_{\mathrm{cone}(\mathcal{P})}\,(1, 1, \ldots, 1, z)$$

Proof of Theorem 4.4

- By Lemma 3.10

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{t \geq 0} L_{\mathcal{P}}(t)\, z^t = \sigma_{\mathrm{cone}(\mathcal{P})}\,(1, 1, \ldots, 1, z)$$

- Similarly

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- Similarly

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = \sum_{t \geq 1} L_{\mathcal{P}^\circ}(t)\, z^t = \sigma_{(\mathrm{cone}(\mathcal{P}))^\circ}(1, 1, \ldots, 1, z)$$

- By applying Stanley's reciprocity (Thm 4.3) to cone($\mathcal{P}$) we obtain

$$\sigma_{(\mathrm{cone}(\mathcal{P}))^\circ}(1, 1, \ldots, 1, z) = (-1)^{d+1}\, \sigma_{\mathrm{cone}(\mathcal{P})}\left(1, 1, \ldots, 1, \frac{1}{z}\right)$$

□

Proof of Ehrhart–Macdonald's reciprocity (Thm 4.1)

- By Exer 4.6

$$-\sum_{t\geq 1} L_{\mathcal{P}}(-t)\, z^t = \sum_{t\leq 0} L_{\mathcal{P}}(-t)\, z^t$$

Proof of Ehrhart–Macdonald's reciprocity (Thm 4.1)

- By Exer 4.6

$$-\sum_{t\geq 1} L_{\mathcal{P}}(-t)\,z^t = \sum_{t\leq 0} L_{\mathcal{P}}(-t)\,z^t$$

$$= \sum_{t\geq 0} L_{\mathcal{P}}(t)\,z^{-t}$$

Proof of Ehrhart–Macdonald's reciprocity (Thm 4.1)

- By Exer 4.6

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- Then by Theorem 4.4

$$\sum_{t\geq 1} L_{\mathcal{P}^\circ}(t)\, z^t = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right)$$

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$$\sum_{t\geq 1} L_{\mathcal{P}^\circ}(t)\, z^t = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^d \sum_{t\geq 1} L_{\mathcal{P}}(-t)\, z^t$$

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- Comparing the coefficients, we obtain the theorem □

Degression: the degree of an integral polytope

Definition (Degree of an integral polytope)

For an integral d-polytope $\mathcal{P}$ with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}},$$

the degree of $\mathcal{P}$ is the largest k s.t. $h_k \neq 0$

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Theorem 4.5

The degree of an integral d-polytope $\mathcal{P}$ is $k \Leftrightarrow$ $(d-k+1)\mathcal{P}$ is the smallest integer dilate of $\mathcal{P}$ that contains an interior lattice point

Proof of Theorem 4.5

- We use Theorem 3.18

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- The degree of $\mathcal{P}$ is $k \Leftrightarrow$
  $L_{\mathcal{P}}(-1) = L_{\mathcal{P}}(-2) = \cdots = L_{\mathcal{P}}\left(-(d-k)\right) = 0$ and
  $L_{\mathcal{P}}\left(-(d-k+1)\right) \neq 0$

Proof of Theorem 4.5

- We use Theorem 3.18
- The degree of $\mathcal{P}$ is $k \Leftrightarrow$
  $L_{\mathcal{P}}(-1) = L_{\mathcal{P}}(-2) = \cdots = L_{\mathcal{P}}(-(d-k)) = 0$ and
  $L_{\mathcal{P}}(-(d-k+1)) \neq 0$
- By the Ehrhart–Macdonald reciprocity, this is equivalent to
  $\mathcal{P}^\circ, (2\mathcal{P})^\circ, \ldots, ((d-k)\mathcal{P})^\circ$ contains no lattice point and
  $((d-k+1)\mathcal{P})^\circ$ contains a lattice point □

Reflexive polytopes

Definition (Reflexive polytope)

A polytope $\mathcal{P}$ that contains the origin in its interior is reflexive if it is integral and has the hyperplane description

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{1} \right\},$$

where $\mathbf{A}$ is an integral matrix

Example: d-Crosspolytopes

Palindromy of the Ehrhart series of a reflexive polytope

Theorem 4.6 (Hibi's palindromic theorem)

$\mathcal{P}$ an integral d-polytope that contains the origin in its interior and that has the Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}}$$

$\Rightarrow$ $\mathcal{P}$ reflexive if and only if $h_k = h_{d-k}\ \forall\ 0 \le k \le \frac{d}{2}$

Example:

$$\mathrm{Ehr}_{\Diamond}(z) = \frac{\sum_{k=0}^{d} \binom{d}{k} z^k}{(1-z)^{d+1}}, \quad \binom{d}{k} = \binom{d}{d-k}$$

A lemma we use for the proof of Thm 4.6

Lemma 4.7

$a_1, a_2, \ldots, a_d, b \in \mathbb{Z}$ satisfy $\gcd(a_1, a_2, \ldots, a_d, b) = 1$ and $b > 1$
$\Rightarrow \exists\ c, t \in \mathbb{Z}_{>0}$ s.t.

- $tb < c < (t+1)b$,
- $\{(m_1, \ldots, m_d) \in \mathbb{Z}^d : a_1 m_1 + \cdots + a_d m_d = c\} \neq \emptyset$

Proof:

- Let $g = \gcd(a_1, a_2, \ldots, a_d)$

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- Let $g = \gcd(a_1, a_2, \ldots, a_d)$
- $\exists\ k \in \mathbb{Z}$ and $t \in \mathbb{Z}_{>0}$ s.t. $kg - tb = 1$ $\quad (\because \gcd(g, b) = 1)$

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- Let $c = kg$

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- Then $tb < c < (t+1)b$ $\quad (\because kg - tb = 1, b > 1)$

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- $\exists\ k \in \mathbb{Z}$ and $t \in \mathbb{Z}_{>0}$ s.t. $kg - tb = 1$ $\quad (\because \gcd(g, b) = 1)$
- Let $c = kg$
- Then $tb < c < (t+1)b$ $\quad (\because kg - tb = 1, b > 1)$
- Since $g = \gcd(a_1, a_2, \ldots, a_d)$, $\exists\ m_1, m_2, \ldots, m_d \in \mathbb{Z}$ s.t.

$$a_1 m_1 + a_2 m_2 + \cdots + a_d m_d = kg = c \qquad \square$$

Proof of Thm 4.6

Claim

$\mathcal{P}$ reflexive $\Leftrightarrow$

- $\mathcal{P}^\circ \cap \mathbb{Z}^d = \{\mathbf{0}\}$
- $(t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d = t\mathcal{P} \cap \mathbb{Z}^d$ for all $t \in \mathbb{Z}_{>0}$

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Proof of $\Rightarrow$: Exercise 4.12

Proof of Thm 4.6

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Proof of $\Rightarrow$: Exercise 4.12

Proof of $\Leftarrow$: Assume $\mathcal{P}$ satisfies the conditions on the RHS

Proof of Thm 4.6

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Proof of $\Rightarrow$: Exercise 4.12

Proof of $\Leftarrow$: Assume $\mathcal{P}$ satisfies the conditions on the RHS

- Let $H = \left\{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d = b\right\}$ define a facet of $\mathcal{P}$ (wlog $\gcd(a_1, a_2, \ldots, a_d, b) = 1$)

Proof of Thm 4.6

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- $\exists$ no lattice point between tH and $(t+1)H$ (by Exer 4.13)

Proof of Thm 4.6

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- $\exists$ no lattice point between tH and $(t+1)H$ (by Exer 4.13)
- $\therefore \left\{\mathbf{x} \in \mathbb{Z}^d : tb < a_1x_1 + a_2x_2 + \cdots + a_dx_d < (t+1)b\right\} = \varnothing$

Proof of Thm 4.6

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- $\mathcal{P}^\circ \cap \mathbb{Z}^d = \{\mathbf{0}\}$
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- $\therefore \left\{\mathbf{x} \in \mathbb{Z}^d : tb < a_1x_1 + a_2x_2 + \cdots + a_dx_d < (t+1)b\right\} = \varnothing$
- $\therefore b = 1$ (by Lem 4.7)

□

Proof of Thm 4.6 (cont'd)

- By Theorem 4.4

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right)$$

Proof of Thm 4.6 (cont'd)

- By Theorem 4.4

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = \frac{h_0\, z^{d+1} + h_1\, z^d + \cdots + h_{d-1}\, z^2 + h_d\, z}{(1-z)^{d+1}}$$

Proof of Thm 4.6 (cont'd)

- By Theorem 4.4

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = \frac{h_0\,z^{d+1} + h_1\,z^{d} + \cdots + h_{d-1}\,z^2 + h_d\,z}{(1-z)^{d+1}}$$

- By Claim, $\mathcal{P}$ reflexive iff $\mathrm{Ehr}_{\mathcal{P}^\circ}(z)$ is equal to

$$\sum_{t \geq 1} L_{\mathcal{P}}(t-1)\,z^t$$

□

Proof of Thm 4.6 (cont'd)

- By Theorem 4.4

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- By Claim, $\mathcal{P}$ reflexive iff $\mathrm{Ehr}_{\mathcal{P}^\circ}(z)$ is equal to

$$\sum_{t\geq 1} L_{\mathcal{P}}(t-1)\,z^t = z\sum_{t\geq 0} L_{\mathcal{P}}(t)\,z^t$$

□

Proof of Thm 4.6 (cont'd)

- By Theorem 4.4

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}^\circ}(z) &= (-1)^{d+1}\,\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right)\\ &= \frac{h_0\,z^{d+1} + h_1\,z^d + \cdots + h_{d-1}\,z^2 + h_d\,z}{(1-z)^{d+1}}\end{aligned}$$

- By Claim, $\mathcal{P}$ reflexive iff $\mathrm{Ehr}_{\mathcal{P}^\circ}(z)$ is equal to

$$\sum_{t\geq 1} L_{\mathcal{P}}(t-1)\,z^t = z\sum_{t\geq 0} L_{\mathcal{P}}(t)\,z^t = z\,\mathrm{Ehr}_{\mathcal{P}}(z)$$

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- By Theorem 4.4

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□

Summary

Theorem 4.1 (Ehrhart–Macdonald reciprocity)

$\mathcal{P}$ a convex rational polytope $\Rightarrow$ for any $t \in \mathbb{Z}_{>0}$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$$

Theorem 4.6 (Hibi's palindromic theorem)

$\mathcal{P}$ an integral d-polytope that contains the origin in its interior and that has the Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}}$$

$\Rightarrow$ $\mathcal{P}$ reflexive if and only if $h_k = h_{d-k}$ $\forall\ 0 \le k \le \frac{d}{2}$

Rest of the course

- Dehn–Sommerville relations
  - Relations among the face numbers of a polytope
  - Ehrhart–Macdonald's reciprocity will be used as a tool
- Magic squares
  - Concrete example of a lattice point counting
  - Ehrhart–Macdonald's reciprocity will be used as a tool
- Finite Fourier series
  - Study of periodic functions
- Fourier–Dedekind sums
  - Appeared in Ehrhart quasipolynomials
  - Computational aspects
- Decomposition of a polytope into cones (Brion's theorem)
  - A magical relation between a polytope and its vertex cones
  - Computational aspects