Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(4) Ehrhart Theory II

Yoshio Okamoto

Tokyo Institute of Technology

May 14, 2009

"Last updated: 2009/05/13 20:40"

From the proof of Ehrhart's theorem

Δ an integral d-simplex, Π the fundamental parallelepiped of cone(Δ)

Consequence of the proof of Ehrhart's theorem

$$\mathrm{Ehr}_\Delta(z) = \frac{\sigma_\Pi(1,\dots,1,z)}{(1-z)^{d+1}}$$

Corollary 3.11

If

$$\mathrm{Ehr}_\Delta(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}},$$

then

$$h_k = \#(\Pi \cap \{\mathbf{x} : x_{d+1} = k\} \cap \mathbb{Z}^{d+1})$$

Comments to Corollary 3.11

Corollary 3.11

If
$$\mathrm{Ehr}_\Delta(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}},$$
then
$$h_k = \#(\Pi \cap \{\mathbf{x} : x_{d+1} = k\} \cap \mathbb{Z}^{d+1})$$

- This enables us to compute $\mathrm{Ehr}_\Delta(z)$ efficiently when d is relatively small
  - But not for a general integral polytope
- The h_k are all nonnegative
  - How about for a general integral polytope?

Stanley's nonnegativity theorem

Theorem 3.12 (Stanley's nonnegativity theorem '80

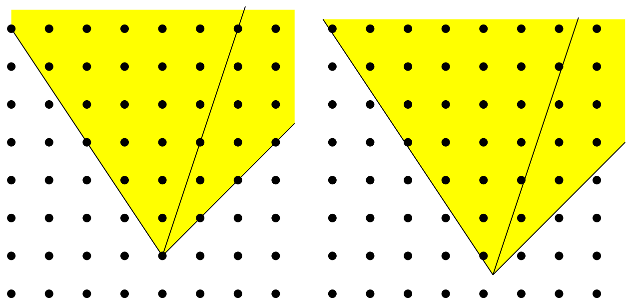
$\mathcal{P}$ an integral convex d-polytope

If
$$\mathrm{Ehr}_\mathcal{P}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_0}{(1-z)^{d+1}}$$
then $h_0, h_1, \ldots, h_d \geq 0$

Remember the examples from Chapter 2!

Proof of Thm 3.12

- Triangulate $\mathrm{cone}(\mathcal{P})$ into simplicial cones $\mathcal{K}_1, \ldots, \mathcal{K}_m$ (Thm 3.1)
- $\exists$ a vector $\mathbf{v} \in \mathbb{R}^{d+1}$ s.t. (Exer 3.14)
  - $\mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^{d+1} = (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1}$ and
  - Neither the facets of $\mathbf{v} + \mathrm{cone}(\mathcal{P})$ nor the triangulation hyperplanes contain any lattice points



Proof of Thm 3.12 (cont'd)

- Then $\forall\, \mathbf{x} \in (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1}\ \exists!\ j \in \{1, \ldots, m\}$: $\mathbf{x} \in \mathbf{v} + \mathcal{K}_j$
- $\therefore$ it holds as a disjoint union
$$\mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^d = (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^d = \bigcup_{j=1}^{m} ((\mathbf{v} + \mathcal{K}_j) \cap \mathbb{Z}^d) \quad (2)$$
- $\therefore$
$$\sigma_{\mathrm{cone}(\mathcal{P})}(z_1, z_2, \ldots, z_{d+1}) = \sum_{j=1}^{m} \sigma_{\mathbf{v}+\mathcal{K}_j}(z_1, z_2, \ldots, z_{d+1})$$
- $\therefore$ by Lemma 3.10
$$\mathrm{Ehr}_\mathcal{P}(z) = \sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \ldots, 1, z) = \sum_{j=1}^{m} \sigma_{\mathbf{v}+\mathcal{K}_j}(1, 1, \ldots, 1, z) \quad (3)$$

Proof of Thm 3.12 (further cont'd)

- Enough to show that each $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, 1, \ldots, 1, z)$ has a nonneg numerator
- The numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, \ldots, 1, z)$ is $\sigma_{\mathbf{v}+\Pi}(1, \ldots, 1, z)$, where Π is the (open) fundamental parallelepiped (Cor. 3.6)
- Each term in $\sigma_{\mathbf{v}+\Pi}(\mathbf{z})$ has a nonnegative exponent in z_{d+1}
  - $\because$ $(\mathbf{v}+\Pi) \cap \mathbb{Z}^{d+1} \subseteq (\mathbf{v}+\mathcal{K}_j) \cap \mathbb{Z}^{d+1} = (\mathbf{v}+\text{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1} = \text{cone}(\mathcal{P}) \cap \mathbb{Z}^{d+1}$
- $\therefore$ The numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, \ldots, 1, z)$ has a nonnegative exponent in z □

Corollary: A constant term

Lemma 3.13

$\mathcal{P}$ an integral convex d-polytope with Ehrhart series

$$\text{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_0}{(1-z)^{d+1}}$$

$\Rightarrow h_0 = 1$

Proof:

- As in Thm 3.12, consider $\mathcal{K}_1, \ldots, \mathcal{K}_m$ and $\mathbf{v}$
- $\exists!\ j \in \{1, \ldots, m\}$: $\mathbf{0} \in \mathbf{v} + \mathcal{K}_j$
- For such a j, the constant term of the numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, \ldots, 1, z)$ is 1
- For the other j, the constant term of the numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, \ldots, 1, z)$ is 0 □

How to extract the Ehrhart polynomial from the Ehrhart series

Lemma 3.14

$\mathcal{P}$ an integral convex d-polytope with Ehrhart series

$$\begin{aligned}\text{Ehr}_{\mathcal{P}}(z) &= 1 + \sum_{t\ge1} L_{\mathcal{P}}(t)\, z^t = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}} \\ \Rightarrow L_{\mathcal{P}}(t) &= \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \\ &\quad \cdots + h_{d-1}\binom{t+1}{d} + h_d \binom{t}{d}\end{aligned}$$

A (unique) expression of $L_{\mathcal{P}}(t)$ by the basis $\binom{t+d}{d}, \ldots, \binom{t+1}{d}, \binom{t}{d}$ (Exer. 3.9)

Proof of Lem 3.14

$$\begin{aligned}\text{Ehr}_{\mathcal{P}}(z) &= \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}} \\ &= \left(h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1\right) \sum_{t\ge0} \binom{t+d}{d} z^t \\ &= h_d \sum_{t\ge0}\binom{t+d}{d} z^{t+d} + h_{d-1}\sum_{t\ge0}\binom{t+d}{d} z^{t+d-1} + \cdots \\ &\quad + h_1 \sum_{t\ge0}\binom{t+d}{d} z^{t+1} + \sum_{t\ge0}\binom{t+d}{d} z^t \quad \square\end{aligned}$$

Proof of Lem 3.14 (cont'd)

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) &= h_d \sum_{t\geq d}\binom{t}{d} z^t + h_{d-1}\sum_{t\geq d-1}\binom{t+1}{d} z^t + \cdots \\ &\quad + h_1 \sum_{t\geq 1}\binom{t+d-1}{d} z^t + \sum_{t\geq 0}\binom{t+d}{d} z^t \\ &= \sum_{t\geq 0}\left(h_d\binom{t}{d} + h_{d-1}\binom{t+1}{d} + \dots \right. \\ &\qquad \left. + h_1\binom{t+d-1}{d} + \binom{t+d}{d}\right) z^t \quad \square\end{aligned}$$

Constant term of an Ehrhart polynomial

Corollary 3.15

$\mathcal{P}$ an integral convex d-polytope $\Rightarrow$ const $L_{\mathcal{P}}(t) = 1$

Proof:

$$L_{\mathcal{P}}(0) = \binom{d}{d} + h_1\binom{d-1}{d} + \cdots + h_{d-1}\binom{1}{d} + h_d\binom{0}{d} = \binom{d}{d} = 1$$

by Lemma 3.14 $\square$

We know $h_0 = 1$. How about $h_1, \dots$?

Corollary 3.16

$\mathcal{P}$ an integral convex d-polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}}.$$

$\Rightarrow h_1 = L_{\mathcal{P}}(1) - d - 1 = \#\left(\mathcal{P}\cap\mathbb{Z}^d\right) - d - 1$

Proof:

$$L_{\mathcal{P}}(1) = \binom{d+1}{d} + h_1\binom{d}{d} + \cdots + h_{d-1}\binom{2}{d} + h_d\binom{1}{d} = d+1+h_1$$

by Lemma 3.14 $\square$

Remark

We may get similar expressions for $h_2, h_3, \dots$ (Exer. 3.10)

How large the coefficients of Ehrhart polynomials are

Corollary 3.17

$\mathcal{P}$ an integral polytope with Ehrhart polynomial
$L_{\mathcal{P}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_1\, t + 1$
$\Rightarrow d!\, c_k \in \mathbb{Z}$ for all k

Proof:

- By Thm 3.12 and Lem 3.14

$$L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1\binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d\binom{t}{d},$$

  where the h_k are integers
- Expanding the binomial coefficients gives a polynomial in t and the coefficient can be written as rational numbers with denominator $d!$ $\square$

This will be used in the next lecture

Theorem 3.18

Let p be a degree-d polynomial with the rational generating function

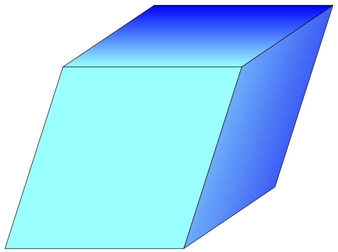
$$\sum_{t \geq 0} p(t)\, z^t = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}};$$

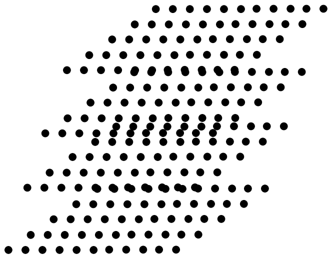
Then

$$\begin{array}{c} h_d = h_{d-1} = \\ \cdots = h_{k+1} = 0 \\ \text{and } h_k \neq 0 \end{array} \Leftrightarrow \begin{array}{c} p(-1) = p(-2) = \\ \cdots = p(-(d-k)) = 0 \\ \text{and } p(-(d-k+1)) \neq 0 \end{array}$$

Proof: Omitted (see the textbook)

What's discrete volume? (from the 1st lecture)





$$\mathrm{vol}\, S = \lim_{t\to\infty} \# \left(S \cap \frac{1}{t}\mathbb{Z}^d \right) \frac{1}{t^d}$$

integration counting

From the discrete to the continuous volume

Since

$$\# \left(S \cap \left(\frac{1}{t}\mathbb{Z} \right)^d \right) = \# \left(tS \cap \mathbb{Z}^d \right),$$

we obtain the following

Lemma 3.19

$S \subset \mathbb{R}^d$ d-dimensional $\Rightarrow$

$$\mathrm{vol}\, S = \lim_{t\to\infty} \frac{1}{t^d} \cdot \# \left(tS \cap \mathbb{Z}^d \right) \quad \square$$

Note: If S is not d-dimensional then $\mathrm{vol}\, S = 0$ by definition

A nice consequence of Ehrhart's theorem

Corollary 3.20

$\mathcal{P} \subset \mathbb{R}^d$ an integral convex d-polytope with Ehrhart polynomial $c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_1\, t + 1 \Rightarrow c_d = \operatorname{vol} \mathcal{P}$

Proof:

$$\operatorname{vol} \mathcal{P} = \lim_{t\to\infty} \frac{c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_1\, t + 1}{t^d} = c_d \quad \square$$

Extracting the continuous volume from the Ehrhart series

Corollary 3.21

$\mathcal{P} \subset \mathbb{R}^d$ an integral convex d-polytope, and

$$\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}}$$

$$\Rightarrow \operatorname{vol} \mathcal{P} = \frac{1}{d!}\,(h_d + h_{d-1} + \cdots + h_1 + 1)$$

Proof: Lemma 3.14 gives

$$L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d \binom{t}{d}$$

and the coefficient of t^d is the desired expression $\square$

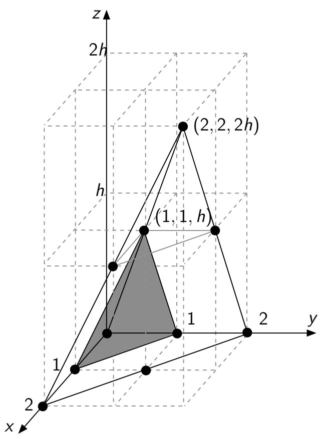
A way to compute the Ehrhart polynomials

How can we compute $L_{\mathcal{P}}(t)$ of a given integral d-polytope $\mathcal{P}$?

- We can make use of Ehrhart's theorem
  - $\mathcal{L}_{\mathcal{P}}(t)$ is a degree-d polynomial in t
- A degree-d polynomial is uniquely determined by the values on $d+1$ points
- Lagrange interpolation: Determining such a unique polynomial from $d+1$ values
  - This involves a famous Vandermonde matrix

Example: Reeve's tetrahedron

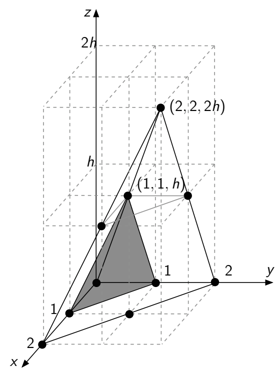
$\mathcal{T}_h$ = the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,h)$, where h is a positive integer



Example: Reeve's tetrahedron (2)

Let $L_{\mathcal{T}_h}(t) = c_3\,t^3 + c_2\,t^2 + c_1\,t + 1$; Then

$$4 = L_{\mathcal{T}_h}(1) = c_3 + c_2 + c_1 + 1$$
$$h + 9 = L_{\mathcal{T}_h}(2) = c_3 \cdot 2^3 + c_2 \cdot 2^2 + c_1 \cdot 2 + 1$$



Example: Reeve's tetrahedron (3)

By Cor 3.20,

$$c_3 = \mathrm{vol}\,(\mathcal{T}_h) = \frac{1}{3}(\text{base area})(\text{height}) = \frac{h}{6}$$

Therefore

$$4 = c_3 + c_2 + c_1 + 1 = \frac{h}{6} + c_2 + c_1 + 1$$
$$h + 9 = c_3 \cdot 2^3 + c_2 \cdot 2^2 + c_1 \cdot 2 + 1 = 8 \cdot \frac{h}{6} + 4\,c_2 + 2\,c_1 + 1$$

Hence $c_2 = 1, c_1 = 2 - \frac{h}{6}$ □

Ehrhart's theorem for rational polytopes

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$

Definition (Ehrhart quasipolynomial)

$L_{\mathcal{P}}$ is called the Ehrhart quasipolynomial of $\mathcal{P}$ when $\mathcal{P}$ is a rational convex polytope

Definition (Denominator of a polytope)

The denominator of $\mathcal{P}$ is the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$

Proof outline

Similar to Ehrhart's theorem (Thm. 3.8)

- Enough to show for simplices Δ (by triangulation)
- See a relation between L_Δ and $\mathrm{Ehr}_\Delta(z)$ (Lem 3.24)
- Go along the same way as in the proof of Thm. 3.8 (Exer 3.20)

Lemma 3.24

Let

$$\sum_{t\geq 0} f(t)\, z^t = \frac{g(z)}{h(z)};$$

Then f is a quasipolynomial of degree d with period dividing p if and only if g and h are polynomials s.t. $\deg(g) < \deg(h)$, all roots of h are pth roots of unity of multiplicity at most $d+1$, and $\exists$ a root of multiplicity equal to $d+1$ (all of this assuming that g/h has been reduced to lowest terms)

Proof: Exercise 3.19

Proof outline (cont'd)

$\therefore$ Enough to prove the following

Claim

Δ a rational d-simplex with denominator $p \Rightarrow$

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t\geq 1} L_{\mathcal{P}}(t)\, z^t = \frac{g(z)}{(1-z^p)^{d+1}}$$

for some polynomial g of degree less than $p(d+1)$

- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1} \in \mathbb{Q}^d$ the vertices of Δ w/ denom p
- Consider cone(Δ) with generators

$$\mathbf{w}_1 = (\mathbf{v}_1, 1), \mathbf{w}_2 = (\mathbf{v}_2, 1), \ldots, \mathbf{w}_{d+1} = (\mathbf{v}_{d+1}, 1)$$

Proof outline (further cont'd)

- We want to use Theorem 3.5
  - But, Thm 3.5 is for integral pointed cones
- However, replacing $\mathbf{w}_k \in \mathbb{Q}^{d+1}$ by $p\mathbf{w}_k \in \mathbb{Z}^{d+1}$ doesn't change cone(Δ)!
- Now the proof goes along the same line as we did for Thm 3.8 (Exer. 3.20)

□