Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(4) Ehrhart Theory II

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1. The Ehrhart Series of an Integral Polytope

2. From the Discrete to the Continuous Volume of a Polytope

3. Interpolation

4. Rational Polytopes and Ehrhart Quasipolynomials

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2. From the Discrete to the Continuous Volume of a Polytope

3. Interpolation

4. Rational Polytopes and Ehrhart Quasipolynomials

From the proof of Ehrhart's theorem

Δ an integral d-simplex, Π the fundamental parallelepiped of cone(Δ)

Consequence of the proof of Ehrhart's theorem

$$\mathrm{Ehr}_\Delta(z) = \frac{\sigma_\Pi(1,\ldots,1,z)}{(1-z)^{d+1}}$$

Corollary 3.11

If

$$\mathrm{Ehr}_\Delta(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}},$$

then

$$h_k = \#(\Pi \cap \{\mathbf{x} : x_{d+1} = k\} \cap \mathbb{Z}^{d+1})$$

Comments to Corollary 3.11

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- This enables us to compute $\mathrm{Ehr}_\Delta(z)$ efficiently when d is relatively small
  - But not for a general integral polytope
- The h_k are all nonnegative
  - How about for a general integral polytope?

Stanley's nonnegativity theorem

Theorem 3.12 (Stanley's nonnegativity theorem '80

$\mathcal{P}$ an integral convex d-polytope
If

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_0}{(1-z)^{d+1}}$$

then $h_0, h_1, \ldots, h_d \geq 0$

Remember the examples from Chapter 2!

Proof of Thm 3.12

- Triangulate cone($\mathcal{P}$) into simplicial cones $\mathcal{K}_1, \ldots, \mathcal{K}_m$ (Thm 3.1)

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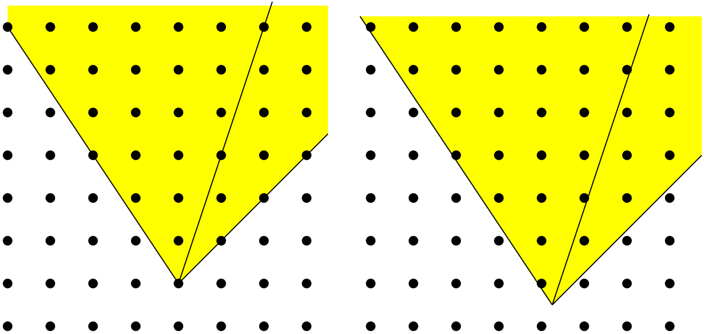
- Triangulate cone($\mathcal{P}$) into simplicial cones $\mathcal{K}_1, \dots, \mathcal{K}_m$ (Thm 3.1)
- $\exists$ a vector $\mathbf{v} \in \mathbb{R}^{d+1}$ s.t. (Exer 3.14)
  - $\text{cone}(\mathcal{P}) \cap \mathbb{Z}^{d+1} = (\mathbf{v} + \text{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1}$ and
  - Neither the facets of $\mathbf{v} + \text{cone}(\mathcal{P})$ nor the triangulation hyperplanes contain any lattice points



Proof of Thm 3.12 (cont'd)

- Then $\forall\, \mathbf{x} \in (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1}\ \exists!\ j \in \{1, \ldots, m\}$: $\mathbf{x} \in \mathbf{v} + \mathcal{K}_j$

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- Then $\forall\, \mathbf{x} \in (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1}\ \exists!\ j \in \{1, \dots, m\}$: $\mathbf{x} \in \mathbf{v} + \mathcal{K}_j$
- $\therefore$ it holds

$$\mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^d = (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^d \tag{2}$$

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- $\therefore$

$$\sigma_{\mathrm{cone}(\mathcal{P})}\left(z_1, z_2, \ldots, z_{d+1}\right) = \sum_{j=1}^{m} \sigma_{\mathbf{v}+\mathcal{K}_j}\left(z_1, z_2, \ldots, z_{d+1}\right)$$

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- $\therefore$

$$\sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \ldots, 1, z) = \sum_{j=1}^{m} \sigma_{\mathbf{v}+\mathcal{K}_j}(1, 1, \ldots, 1, z) \quad (3)$$

Proof of Thm 3.12 (cont'd)

- Then $\forall\, \mathbf{x} \in (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1}\ \exists!\ j \in \{1, \dots, m\}$: $\mathbf{x} \in \mathbf{v} + \mathcal{K}_j$
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- $\therefore$ by Lemma 3.10

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \dots, 1, z) = \sum_{j=1}^{m} \sigma_{\mathbf{v}+\mathcal{K}_j}(1, 1, \dots, 1, z) \quad (3)$$

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- Each term in $\sigma_{\mathbf{v}+\Pi}(\mathbf{z})$ has a nonnegative exponent in z_{d+1}
  - $\because (\mathbf{v}+\Pi) \cap \mathbb{Z}^{d+1} \subseteq (\mathbf{v}+\mathcal{K}_j) \cap \mathbb{Z}^{d+1} = (\mathbf{v}+\mathrm{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1} = \mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^{d+1}$

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- $\therefore$ The numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, \ldots, 1, z)$ has a nonnegative exponent in z $\square$

Corollary: A constant term

Lemma 3.13

$\mathcal{P}$ an integral convex d-polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_0}{(1-z)^{d+1}}$$

$\Rightarrow h_0 = 1$

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- As in Thm 3.12, consider $\mathcal{K}_1, \ldots, \mathcal{K}_m$ and $\mathbf{v}$

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- For such a j, the constant term of the numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, \ldots, 1, z)$ is 1
- For the other j, the constant term of the numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1, \ldots, 1, z)$ is 0

$\square$

How to extract the Ehrhart polynomial from the Ehrhart series

Lemma 3.14

$\mathcal{P}$ an integral convex d-polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}}$$

$$\Rightarrow L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d \binom{t}{d}$$

A (unique) expression of $L_{\mathcal{P}}(t)$ by the basis $\binom{t+d}{d}, \ldots, \binom{t+1}{d}, \binom{t}{d}$ (Exer. 3.9)

Proof of Lem 3.14

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}}$$

Proof of Lem 3.14

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) &= \frac{h_d\,z^d + h_{d-1}\,z^{d-1} + \cdots + h_1\,z + 1}{(1-z)^{d+1}} \\ &= \left(h_d\,z^d + h_{d-1}\,z^{d-1} + \cdots + h_1\,z + 1\right)\sum_{t\geq 0}\binom{t+d}{d} z^t\end{aligned}$$

Proof of Lem 3.14

$$\begin{aligned}
\mathrm{Ehr}_{\mathcal{P}}(z) &= \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}} \\
&= \left(h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1\right) \sum_{t \geq 0} \binom{t+d}{d} z^t \\
&= h_d \sum_{t \geq 0} \binom{t+d}{d} z^{t+d} + h_{d-1} \sum_{t \geq 0} \binom{t+d}{d} z^{t+d-1} + \cdots \\
&\quad + h_1 \sum_{t \geq 0} \binom{t+d}{d} z^{t+1} + \sum_{t \geq 0} \binom{t+d}{d} z^t \qquad \square
\end{aligned}$$

Proof of Lem 3.14 (cont'd)

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) = h_d \sum_{t \geq d} \binom{t}{d} z^t + h_{d-1} \sum_{t \geq d-1} \binom{t+1}{d} z^t + \cdots \\ + h_1 \sum_{t \geq 1} \binom{t+d-1}{d} z^t + \sum_{t \geq 0} \binom{t+d}{d} z^t\end{aligned}$$

Proof of Lem 3.14 (cont'd)

$$
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&\qquad + h_1 \sum_{t \geq 1} \binom{t+d-1}{d} z^t + \sum_{t \geq 0} \binom{t+d}{d} z^t \\
&= \sum_{t \geq 0} \left(h_d \binom{t}{d} + h_{d-1} \binom{t+1}{d} + \ldots \right. \\
&\qquad \left. + h_1 \binom{t+d-1}{d} + \binom{t+d}{d} \right) z^t \quad \square
\end{aligned}
$$

Constant term of an Ehrhart polynomial

Corollary 3.15

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Proof:

$$L_{\mathcal{P}}(0) = \binom{d}{d} + h_1 \binom{d-1}{d} + \cdots + h_{d-1}\binom{1}{d} + h_d \binom{0}{d} = \binom{d}{d} = 1$$

by Lemma 3.14 $\square$

We know $h_0 = 1$. How about $h_1, \ldots$?

Corollary 3.16

$\mathcal{P}$ an integral convex d-polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}}.$$

$\Rightarrow h_1 = L_{\mathcal{P}}(1) - d - 1 = \#\left(\mathcal{P} \cap \mathbb{Z}^d\right) - d - 1$

Remark

We may get similar expressions for $h_2, h_3, \ldots$ (Exer. 3.10)

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<u>Proof</u>:

$$L_{\mathcal{P}}(1) = \binom{d+1}{d} + h_1 \binom{d}{d} + \cdots + h_{d-1}\binom{2}{d} + h_d \binom{1}{d} = d + 1 + h_1$$

by Lemma 3.14 □

Remark

We may get similar expressions for $h_2, h_3, \ldots$ (Exer. 3.10)

How large the coefficients of Ehrhart polynomials are

Corollary 3.17

$\mathcal{P}$ an integral polytope with Ehrhart polynomial
$L_{\mathcal{P}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_1\, t + 1$
$\Rightarrow d!\, c_k \in \mathbb{Z}$ for all k

How large the coefficients of Ehrhart polynomials are

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Proof:

- By Thm 3.12 and Lem 3.14

$$L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d \binom{t}{d},$$

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  where the h_k are integers

- Expanding the binomial coefficients gives a polynomial in t and the coefficient can be written as rational numbers with denominator $d!$ □

This will be used in the next lecture

Theorem 3.18

Let p be a degree-d polynomial with the rational generating function

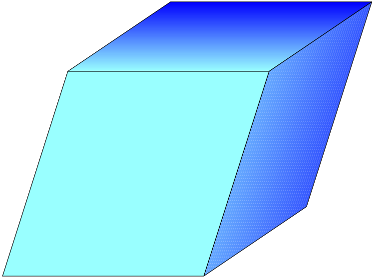
$$\sum_{t \geq 0} p(t)\, z^t = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + h_0}{(1-z)^{d+1}};$$

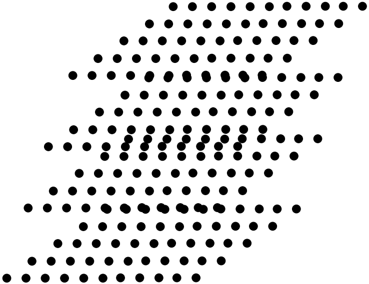
Then

$$\begin{array}{lcl} h_d = h_{d-1} = & & p(-1) = p(-2) = \\ \cdots = h_{k+1} = 0 & \Leftrightarrow & \cdots = p\,(-(d-k)) = 0 \\ \text{and } h_k \neq 0 & & \text{and } p\,(-(d-k+1)) \neq 0 \end{array}$$

<u>Proof</u>: Omitted (see the textbook)

What's discrete volume? (from the 1st lecture)





$$\operatorname{vol} S \quad = \quad \lim_{t\to\infty} \# \left(S \cap \frac{1}{t}\mathbb{Z}^d \right) \frac{1}{t^d}$$

integration counting

From the discrete to the continuous volume

Since

$$\#\left(S \cap \left(\frac{1}{t}\mathbb{Z}\right)^d\right) = \#\left(tS \cap \mathbb{Z}^d\right),$$

we obtain the following

Lemma 3.19

$S \subset \mathbb{R}^d$ d-dimensional $\Rightarrow$

$$\operatorname{vol} S = \lim_{t\to\infty} \frac{1}{t^d} \cdot \#\left(tS \cap \mathbb{Z}^d\right) \quad \square$$

Note: If S is not d-dimensional then $\operatorname{vol} S = 0$ by definition

A nice consequence of Ehrhart's theorem

Corollary 3.20

$\mathcal{P} \subset \mathbb{R}^d$ an integral convex d-polytope with Ehrhart polynomial $c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_1\, t + 1 \Rightarrow c_d = \operatorname{vol} \mathcal{P}$

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Proof:

$$\operatorname{vol} \mathcal{P} = \lim_{t \to \infty} \frac{c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_1\, t + 1}{t^d} = c_d \quad \square$$

Extracting the continuous volume from the Ehrhart series

Corollary 3.21

$\mathcal{P} \subset \mathbb{R}^d$ an integral convex d-polytope, and

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d\, z^d + h_{d-1}\, z^{d-1} + \cdots + h_1\, z + 1}{(1-z)^{d+1}}$$

$$\Rightarrow \mathrm{vol}\,\mathcal{P} = \frac{1}{d!}\,(h_d + h_{d-1} + \cdots + h_1 + 1)$$

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$$\Rightarrow \operatorname{vol} \mathcal{P} = \frac{1}{d!} \left(h_d + h_{d-1} + \cdots + h_1 + 1\right)$$

Proof: Lemma 3.14 gives

$$L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_{d-1} \binom{t+1}{d} + h_d \binom{t}{d}$$

and the coefficient of t^d is the desired expression $\square$

A way to compute the Ehrhart polynomials

How can we compute $L_{\mathcal{P}}(t)$ of a given integral d-polytope $\mathcal{P}$?

- We can make use of Ehrhart's theorem

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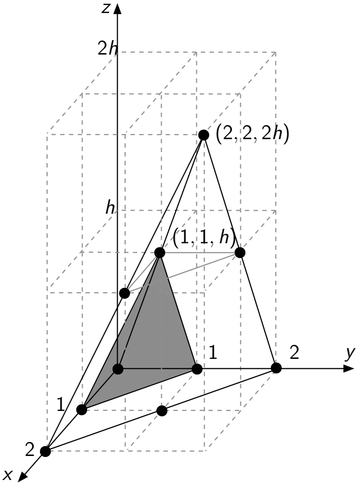
  - $\mathcal{L}_{\mathcal{P}}(t)$ is a degree-d polynomial in t
- A degree-d polynomial is uniquely determined by the values on $d+1$ points
- Lagrange interpolation: Determining such a unique polynomial from $d+1$ values
  - This involves a famous Vandermonde matrix

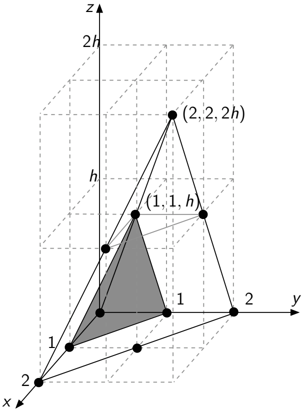
Example: Reeve's tetrahedron

$\mathcal{T}_h$ = the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,h)$, where h is a positive integer



Example: Reeve's tetrahedron (2)

Let $L_{\mathcal{T}_h}(t) = c_3\, t^3 + c_2\, t^2 + c_1\, t + 1$; Then



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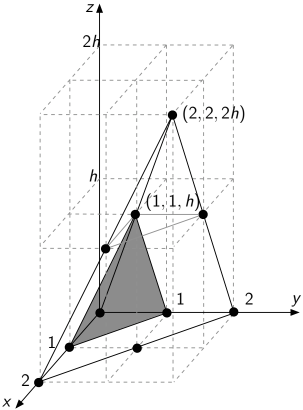

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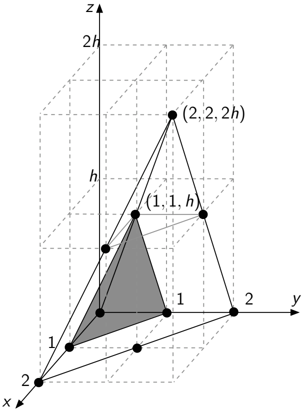
Let $L_{\mathcal{T}_h}(t) = c_3\, t^3 + c_2\, t^2 + c_1\, t + 1$; Then

$$4 = L_{\mathcal{T}_h}(1) = c_3 + c_2 + c_1 + 1$$



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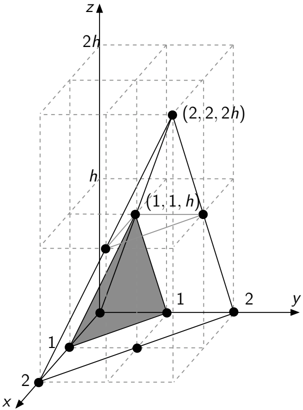

Example: Reeve's tetrahedron (3)

By Cor 3.20,

$$c_3 = \mathrm{vol}\,(\mathcal{T}_h) = \frac{1}{3}(\text{base area})(\text{height}) = \frac{h}{6}$$

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Therefore

$$4 = c_3 + c_2 + c_1 + 1 = \frac{h}{6} + c_2 + c_1 + 1$$

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Hence $c_2 = 1, c_1 = 2 - \frac{h}{6}$ □

Ehrhart's theorem for rational polytopes

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$

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Definition (Ehrhart quasipolynomial)

$L_{\mathcal{P}}$ is called the Ehrhart quasipolynomial of $\mathcal{P}$ when $\mathcal{P}$ is a rational convex polytope

Definition (Denominator of a polytope)

The denominator of $\mathcal{P}$ is the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$

Proof outline

Similar to Ehrhart's theorem (Thm. 3.8)

- Enough to show for simplices Δ (by triangulation)
- See a relation between L_Δ and $\mathrm{Ehr}_\Delta(z)$ (Lem 3.24)
- Go along the same way as in the proof of Thm. 3.8 (Exer 3.20)

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Lemma 3.24

Let

$$\sum_{t \geq 0} f(t)\, z^t = \frac{g(z)}{h(z)}\,;$$

Then f is a quasipolynomial of degree d with period dividing p if and only if g and h are polynomials s.t. $\deg(g) < \deg(h)$, all roots of h are pth roots of unity of multiplicity at most $d+1$, and $\exists$ a root of multiplicity equal to $d+1$ (all of this assuming that g/h has been reduced to lowest terms)

Proof: Exercise 3.19

Proof outline (cont'd)

$\therefore$ Enough to prove the following

Claim

Δ a rational d-simplex with denominator $p \Rightarrow$

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t = \frac{g(z)}{(1 - z^p)^{d+1}}$$

for some polynomial g of degree less than $p(d+1)$

Proof outline (cont'd)

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Claim

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- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1} \in \mathbb{Q}^d$ the vertices of Δ w/ denom p
- Consider cone(Δ) with generators

$$\mathbf{w}_1 = (\mathbf{v}_1, 1), \mathbf{w}_2 = (\mathbf{v}_2, 1), \ldots, \mathbf{w}_{d+1} = (\mathbf{v}_{d+1}, 1)$$

Proof outline (further cont'd)

- We want to use Theorem 3.5
    - But, Thm 3.5 is for integral pointed cones
- However, replacing $\mathbf{w}_k \in \mathbb{Q}^{d+1}$ by $p\mathbf{w}_k \in \mathbb{Z}^{d+1}$ doesn't change cone(Δ)!
- Now the proof goes along the same line as we did for Thm 3.8 (Exer. 3.20)

$\square$