Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(3) Ehrhart Theory I

Yoshio Okamoto

Tokyo Institute of Technology

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1. Triangulations and Pointed Cones

2. Integer-Point Transforms for Rational Cones

3. Expanding and Counting Using Ehrhart's Original Approach

Proving the following two theorems, and some more

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Theorem 3.23 (Ehrhart's Theorem for rational polytopes)

$\mathcal{P}$ is a rational convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d;
Its period divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$

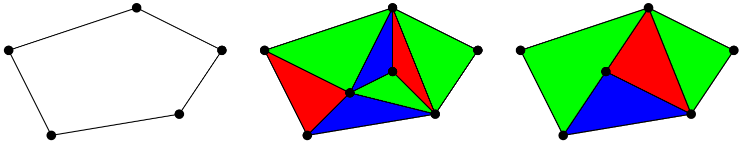
Triangulation

$\mathcal{P}$ a convex d-polytope

Definition (Triangulation)

A triangulation of $\mathcal{P}$ is a finite collection T of d-simplices with the following properties:

- $\mathcal{P} = \bigcup_{\Delta \in T} \Delta$
- $\forall\ \Delta_1, \Delta_2 \in T$: $\Delta_1 \cap \Delta_2$ is a face common to both Δ_1 and Δ_2



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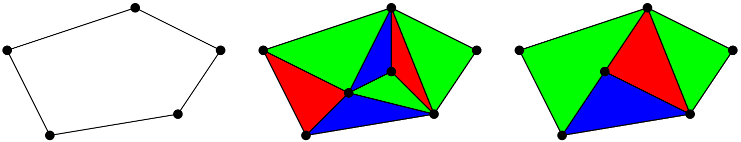
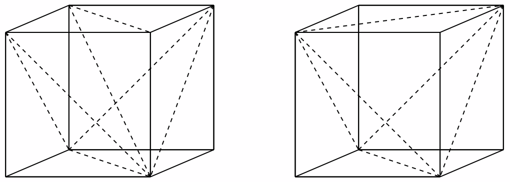


↑not a triangulation

Triangulation using no new vertices

Definition (Triangulation using no new vertices)

$\mathcal{P}$ can be triangulated using no new vertices if $\exists$ a triangulation T s.t. the vertices of any $\Delta \in T$ are vertices of $\mathcal{P}$



Theorem 3.1

Every convex polytope can be triangulated using no new vertices

Proof: See Appendix B in the textbook

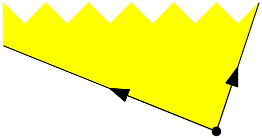
Pointed cones

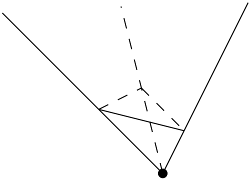
Definition (Pointed cone)

A pointed cone $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_m \mathbf{w}_m : \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\},$$

where $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{R}^d$ are such that $\exists$ a hyperplane H for which $H \cap \mathcal{K} = \{\mathbf{v}\}$; that is, $\mathcal{K} \setminus \{\mathbf{v}\}$ lies strictly on one side of H





Pointed cones: Glossary

A pointed cone
$\mathcal{K} = \{\mathbf{v} + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_m\mathbf{w}_m : \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\} \subseteq \mathbb{R}^d$

Definition

- The vector $\mathbf{v}$ is called the apex of $\mathcal{K}$
- The $\mathbf{w}_k$'s are the generators of $\mathcal{K}$
- The dimension of $\mathcal{K}$ is the dimension of the affine space spanned by $\mathcal{K}$; if $\mathcal{K}$ is of dimension d, we call it a d-cone
- The d-cone $\mathcal{K}$ is simplicial if $\mathcal{K}$ has precisely d linearly independent generators
- The cone is rational if $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{Q}^d$, in which case we may choose $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{Z}^d$ by clearing denominators

Coning over a polytope

$\mathcal{P} \subset \mathbb{R}^d$ a convex polytope with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

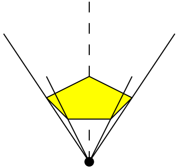
Definition (Cone over a polytope)

The cone over $\mathcal{P}$ is defined as

$$\text{cone}(\mathcal{P}) = \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_n \mathbf{w}_n : \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\} \subset \mathbb{R}^{d+1},$$

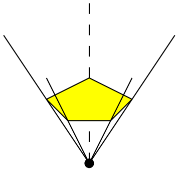
where

$$\mathbf{w}_1 = (\mathbf{v}_1, 1), \ \mathbf{w}_2 = (\mathbf{v}_2, 1), \ \ldots, \ \mathbf{w}_n = (\mathbf{v}_n, 1)$$



Properties of the cone over a polytope

- cone($\mathcal{P}$) has the origin as apex
- We can recover our original polytope $\mathcal{P}$ by cutting cone($\mathcal{P}$) with the hyperplane $x_{d+1} = 1$

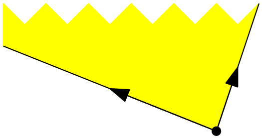


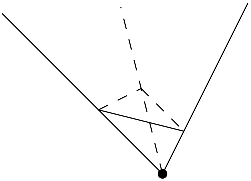
Valid inequalities: Analogous to polytopes

$\mathcal{K} \subseteq \mathbb{R}^d$ a pointed d-cone; $\mathbf{a} \in \mathbb{R}^d$, $b \in \mathbb{R}$

Definition (Valid inequality)

The inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ is a valid inequality for $\mathcal{K}$ if $\mathbf{a} \cdot \mathbf{z} \leq b$ for all $\mathbf{z} \in \mathcal{K}$





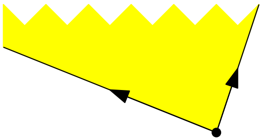
Faces of a pointed cone: Analogous to polytopes

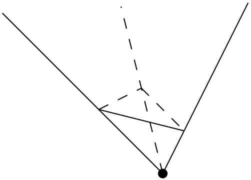
$\mathcal{K} \subseteq \mathbb{R}^d$ a pointed cone

Definition (Face)

$\mathcal{F}$ is a face of $\mathcal{K}$ if $\exists$ a valid inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ for $\mathcal{K}$ s.t.

$$\mathcal{F} = \mathcal{K} \cap \{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = b\}$$





Remark

- Every face of a pointed cone is also a pointed cone

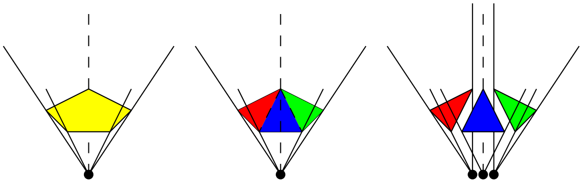
Triangulation of a pointed cone: Analogous to polytopes

$\mathcal{K}$ a pointed d-cone

Definition (Triangulation)

A triangulation of $\mathcal{K}$ is a collection T of simplicial d-cones that satisfies the following:

- $\mathcal{K} = \bigcup_{\mathcal{S} \in T} \mathcal{S}$
- $\forall\ \mathcal{S}_1, \mathcal{S}_2 \in T$: $\mathcal{S}_1 \cap \mathcal{S}_2$ is a face common to both $\mathcal{S}_1$ and $\mathcal{S}_2$



Triangulation using no new generators

$\mathcal{K}$ a pointed d-cone

Definition

$\mathcal{K}$ is triangulated using no new generators if $\exists$ a triangulation T s.t. the generators of any $\mathcal{S} \in T$ are generators of $\mathcal{P}$

Theorem 3.2

Any pointed cone can be triangulated into simplicial cones using no new generators

Proof of Theorem 3.2

- $\mathcal{K}$ a given pointed d-cone
- $\exists$ a hyperplane H that intersects $\mathcal{K}$ only at the apex

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- This intersection is a $(d-1)$-polytope $\mathcal{P}$, whose vertices are determined by the generators of $\mathcal{K}$

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- Triangulate $\mathcal{P}$ using no new vertices (by Thm 3.1)

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- This intersection is a $(d-1)$-polytope $\mathcal{P}$, whose vertices are determined by the generators of $\mathcal{K}$
- Triangulate $\mathcal{P}$ using no new vertices (by Thm 3.1)
- The cone over each simplex of the triangulation is a simplicial cone
- These simplicial cones, by construction, triangulate $\mathcal{K}$ $\square$

Integer-point transforms

Definition (Integer-point transform)

The integer-point transform (or the moment generating function) of $S \subseteq \mathbb{R}^d$ is

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \ldots, z_d) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$$

Recall: $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

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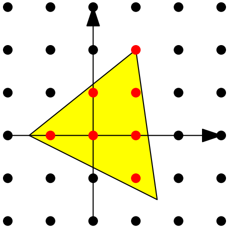


Example:

$$\sigma_S(z_1, z_2) = z_1 z_2^2 + z_1 z_2 + z_1 + z_1 z_2^{-1} + z_2 + 1 + z_1^{-1}$$

Example (1)

$\mathcal{K} = [0, \infty)$ the 1-dimensional cone

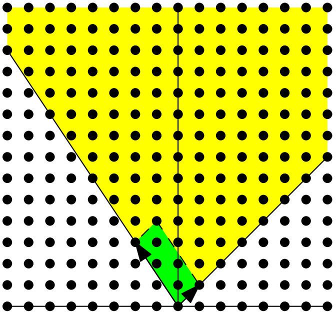
$$\sigma_{\mathcal{K}}(z) = \sum_{m \in [0,\infty) \cap \mathbb{Z}} z^m = \sum_{m \geq 0} z^m = \frac{1}{1-z}$$

Example (2)

$\mathcal{K} := \{\lambda_1(1,1) + \lambda_2(-2,3) : \lambda_1, \lambda_2 \geq 0\} \subset \mathbb{R}^2$;
The fundamental parallelogram of $\mathcal{K}$

$$\Pi := \{\lambda_1(1,1) + \lambda_2(-2,3) : 0 \leq \lambda_1, \lambda_2 < 1\} \subset \mathbb{R}^2$$

tiles $\mathcal{K}$ if we translate Π by nonnegative integer linear combinations of the generators $(1,1)$ and $(-2,3)$



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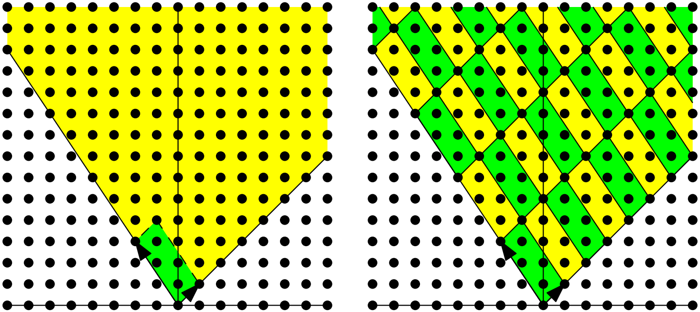
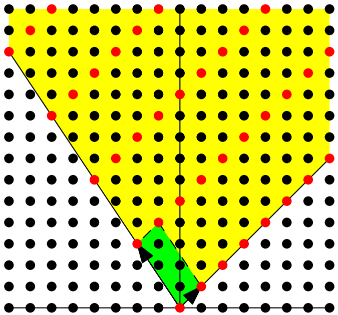


Example (2): List all vertices of the translates of Π

These are nonnegative integer combinations of the generators $(1,1)$ and $(-2,3)$, so we can list them using geometric series:

$$\sum_{\substack{\mathbf{m}=j(1,1)+k(-2,3)\\ j,k\geq 0}} \mathbf{z}^{m} = \sum_{j\geq 0}\sum_{k\geq 0} \mathbf{z}^{j(1,1)+k(-2,3)} = \frac{1}{(1-z_1 z_2)\left(1-z_1^{-2} z_2^{3}\right)}$$



Example (2): Expressing the whole cone by translations

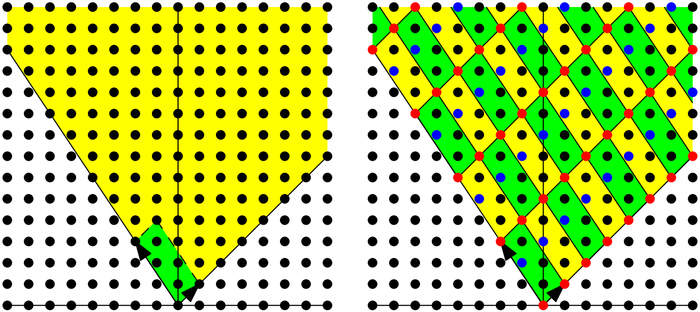
Let

$$\mathcal{L}_{(m,n)} := \{(m,n) + j(1,1) + k(-2,3) : j,k \in \mathbb{Z}_{\geq 0}\}.$$

Then

$$\mathcal{K} \cap \mathbb{Z}^2 = \bigcup_{(m,n)\in \Pi\cap\mathbb{Z}^2} \mathcal{L}_{(m,n)}$$

where $\Pi \cap \mathbb{Z}^2 = \{(0,0),(0,1),(0,2),(-1,2),(-1,3)\}$

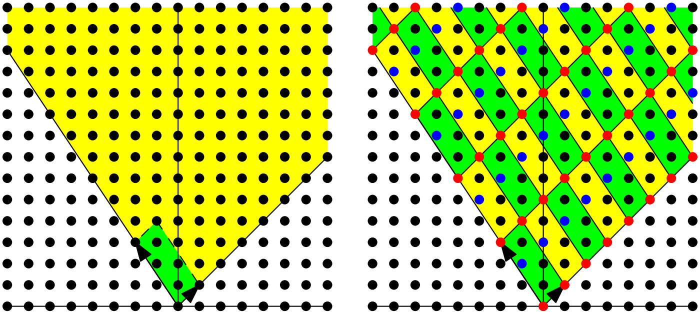


Example (2): Conclusion

Hence

$$\sigma_{\mathcal{K}}(\mathbf{z}) = \left(1 + z_2 + z_2^2 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3\right) \sum_{\substack{\mathbf{m}=j(1,1)+k(-2,3) \\ j,k \geq 0}} \mathbf{z}^m$$

$$= \frac{1 + z_2 + z_2^2 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3}{\left(1 - z_1 z_2\right)\left(1 - z_1^{-2} z_2^3\right)} \quad \square$$



Integer-point transform of a simplicial cone

Theorem 3.5

Let

$$\mathcal{K} := \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : \lambda_1, \lambda_2, \ldots, \lambda_d \geq 0\}$$

be a simplicial d-cone, where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$. Then for $\mathbf{v} \in \mathbb{R}^d$, the integer-point transform $\sigma_{\mathbf{v}+\mathcal{K}}$ of the shifted cone $\mathbf{v} + \mathcal{K}$ is the rational function

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_d})},$$

where Π is the fundamental parallelepiped of $\mathcal{K}$:

$$\Pi := \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_d < 1\}.$$

Proof of Theorem 3.5

- $\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \sum_{\mathbf{m}\in(\mathbf{v}+\mathcal{K})\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$ lists each integer point $\mathbf{m} \in \mathbf{v} + \mathcal{K}$ as the monomial $\mathbf{z}^{\mathbf{m}}$

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- Such a lattice point can be written as (by definition)

$$\mathbf{m} = \mathbf{v} + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_d\mathbf{w}_d$$

for some numbers $\lambda_1, \lambda_2, \ldots, \lambda_d \geq 0$

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  for some numbers $\lambda_1, \lambda_2, \ldots, \lambda_d \geq 0$
- This representation is unique ($\because$ the $\mathbf{w}_k$'s form a basis of $\mathbb{R}^d$)
- Since $\lambda_k = \lfloor\lambda_k\rfloor + \{\lambda_k\}$, we get

$$\begin{aligned}\mathbf{m} &= \mathbf{v} + \left(\{\lambda_1\}\mathbf{w}_1 + \{\lambda_2\}\mathbf{w}_2 + \cdots + \{\lambda_d\}\mathbf{w}_d\right) \\ &\quad + \lfloor\lambda_1\rfloor\mathbf{w}_1 + \lfloor\lambda_2\rfloor\mathbf{w}_2 + \cdots + \lfloor\lambda_d\rfloor\mathbf{w}_d\end{aligned}$$

Proof of Theorem 3.5 (cont'd)

- Since $0 \leq \{\lambda_k\} < 1$,

$$\mathbf{p} := \mathbf{v} + \{\lambda_1\}\,\mathbf{w}_1 + \{\lambda_2\}\,\mathbf{w}_2 + \cdots + \{\lambda_d\}\,\mathbf{w}_d \in \mathbf{v} + \Pi$$

Proof of Theorem 3.5 (cont'd)

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- In fact, $\mathbf{p} \in \mathbb{Z}^d$ ($\because$ $\mathbf{m}$ and $\lfloor \lambda_k \rfloor \mathbf{w}_k$ are all integral)

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- Again the representation of $\mathbf{p}$ in terms of the $\mathbf{w}_k$'s is unique

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- Again the representation of $\mathbf{p}$ in terms of the $\mathbf{w}_k$'s is unique
- $\therefore$ any $\mathbf{m} \in \mathbf{v} + \mathcal{K} \cap \mathbb{Z}^d$ can be uniquely written as

$$\mathbf{m} = \mathbf{p} + k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_d\mathbf{w}_d$$

for some $\mathbf{p} \in (\mathbf{v} + \Pi) \cap \mathbb{Z}^d$ and some $k_1, k_2, \ldots, k_d \in \mathbb{Z}_{\ge 0}$

Proof of Theorem 3.5 (cont'd)

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  for some $\mathbf{p} \in (\mathbf{v} + \Pi) \cap \mathbb{Z}^d$ and some $k_1, k_2, \ldots, k_d \in \mathbb{Z}_{\geq 0}$
- Namely,

$$\begin{aligned} \sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) &= \sum_{\mathbf{m} \in (\mathbf{v}+\mathcal{K}) \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}} \\ &= \sum_{\mathbf{p} \in (\mathbf{v}+\Pi) \cap \mathbb{Z}^d} \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \mathbf{z}^{\mathbf{p} + k_1\mathbf{w}_1 + \cdots + k_d\mathbf{w}_d} \end{aligned}$$

Proof of Theorem 3.5 (further cont'd)

- On the other hand, the RHS of the theorem can be written as

$$\frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})\cdots(1-\mathbf{z}^{\mathbf{w}_d})} = \left(\sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}\right)\left(\sum_{k_1\geq 0} \mathbf{z}^{k_1\mathbf{w}_1}\right)\cdots\left(\sum_{k_d\geq 0} \mathbf{z}^{k_d\mathbf{w}_d}\right)$$

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- On the other hand, the RHS of the theorem can be written as

$$\frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})\cdots(1-\mathbf{z}^{\mathbf{w}_d})} = \left(\sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}\right)\left(\sum_{k_1\geq 0} \mathbf{z}^{k_1\mathbf{w}_1}\right)\cdots\left(\sum_{k_d\geq 0} \mathbf{z}^{k_d\mathbf{w}_d}\right)$$

$$= \sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d} \sum_{k_1\geq 0}\cdots\sum_{k_d\geq 0} \mathbf{z}^{\mathbf{p}+k_1\mathbf{w}_1+\cdots+k_d\mathbf{w}_d} \qquad \square$$

Proof of Theorem 3.5 (further cont'd)

- On the other hand, the RHS of the theorem can be written as

$$\frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})\cdots(1-\mathbf{z}^{\mathbf{w}_d})} = \left(\sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}\right)\left(\sum_{k_1\geq 0} \mathbf{z}^{k_1\mathbf{w}_1}\right)\cdots\left(\sum_{k_d\geq 0} \mathbf{z}^{k_d\mathbf{w}_d}\right)$$
$$= \sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d}\ \sum_{k_1\geq 0}\cdots\sum_{k_d\geq 0} \mathbf{z}^{\mathbf{p}+k_1\mathbf{w}_1+\cdots+k_d\mathbf{w}_d} \qquad \square$$

Remarks

- Crucial geometric idea: $\mathbf{v}+\mathcal{K}$ is tiled with the translates of $\mathbf{v}+\Pi$ by nonnegative integral combinations of the $\mathbf{w}_k$'s
- Computational perspective: Difficulty lies in $\mathbf{v}+\Pi$

Corollary: Relaxing the assumption

Corollary 3.6

Let

$$\mathcal{K} := \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : \lambda_1, \lambda_2, \ldots, \lambda_d \geq 0\}$$

be a simplicial d-cone, where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$, and $\mathbf{v} \in \mathbb{R}^d$, s.t. the boundary of $\mathbf{v} + \mathcal{K}$ contains no integer point. Then

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_d})},$$

where Π is the open parallelepiped

$$\Pi := \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : 0 < \lambda_1, \lambda_2, \ldots, \lambda_d < 1\}.$$

Proof: Similar to Theorem 3.5 □

Corollary: General pointed cones

Corollary 3.7

Given any pointed cone

$$\mathcal{K} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_m \mathbf{w}_m : \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\}$$

with $\mathbf{v} \in \mathbb{R}^d$, $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{Z}^d$, the integer-point transform $\sigma_{\mathcal{K}}(\mathbf{z})$ evaluates to a rational function in the coordinates of $\mathbf{z}$

Proof:

- $\mathcal{K}$ can be triangulated (Theorem 3.2)
- The intersection of simplicial cones in a triangulation is again a simplicial cone (Exer. 3.2)
- The inclusion-exclusion principle does the job □

Ehrhart's Theorem

The fundamental theorem concerning the lattice-point count in an integral convex polytope

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Ehrhart's Theorem

The fundamental theorem concerning the lattice-point count in an integral convex polytope

Theorem 3.8 (Ehrhart's Theorem)

$\mathcal{P}$ is an integral convex d-polytope $\Rightarrow$
$L_{\mathcal{P}}(t)$ is a polynomial in t of degree d

Definition (Ehrhart polynomial)

$L_{\mathcal{P}}$ is called the Ehrhart polynomial of $\mathcal{P}$ when $\mathcal{P}$ is an integral convex polytope

Proof Outline

- Enough to show for simplices Δ (by triangulation)
- See a relation between L_Δ and $\mathrm{Ehr}_\Delta(z)$ (Lem 3.9)
- See a relation between Ehr_Δ and $\sigma_{\mathrm{cone}(\Delta)}$ (Lem 3.10)
- Use Theorem 3.5 to conclude

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- Use Theorem 3.5 to conclude

Lemma 3.9

Let

$$\sum_{t \geq 0} f(t)\, z^t = \frac{g(z)}{(1-z)^{d+1}}\, ;$$

Then f is a polynomial of degree $d \Leftrightarrow g$ is a polynomial of degree at most d and $g(1) \neq 0$

Proof: Exercise 3.8

Proof of Theorem 3.8

- Enough to show for simplices ($\because$ Thm 3.1)

Proof of Theorem 3.8

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$$\mathrm{Ehr}_\Delta(z) = 1 + \sum_{t \geq 1} L_\Delta(t)\, z^t = \frac{g(z)}{(1-z)^{d+1}}$$

for some polynomial g of degree at most d with $g(1) \neq 0$
($\because$ Lem 3.9)

Proof of Theorem 3.8: Lemma 3.10

$\mathcal{P}$ a convex d-polytope

Lemma 3.10

$$\sigma_{\text{cone}(\mathcal{P})}(1, 1, \ldots, 1, z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t = \text{Ehr}_{\mathcal{P}}(z)$$

Proof:

$$\sigma_{\text{cone}(\mathcal{P})}(z_1, z_2, \ldots, z_{d+1})$$

Proof of Theorem 3.8: Lemma 3.10

$\mathcal{P}$ a convex d-polytope

Lemma 3.10

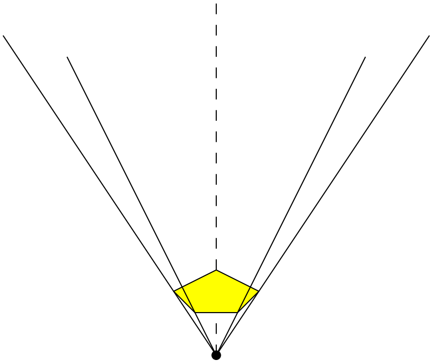
$$\sigma_{\mathrm{cone}(\mathcal{P})}(1,1,\dots,1,z) = 1 + \sum_{t\geq 1} L_{\mathcal{P}}(t)\, z^t = \mathrm{Ehr}_{\mathcal{P}}(z)$$

Proof:

$$\begin{aligned}\sigma_{\mathrm{cone}(\mathcal{P})}&(z_1, z_2, \dots, z_{d+1}) \\ &= 1 + \sigma_{\mathcal{P}}(z_1,\dots,z_d)\, z_{d+1} + \sigma_{2\mathcal{P}}(z_1,\dots,z_d)\, z_{d+1}^2 + \cdots \\ &= 1 + \sum_{t\geq 1} \sigma_{t\mathcal{P}}(z_1,\dots,z_d)\, z_{d+1}^t\end{aligned}$$

Proof of Lemma 3.10 (cont'd)

$$\sigma_{\mathrm{cone}(\mathcal{P})}(\mathbf{z}, z_{d+1}) = 1 + \sigma_{\mathcal{P}}(\mathbf{z})\, z_{d+1} + \sigma_{2\mathcal{P}}(\mathbf{z})\, z_{d+1}^2 + \cdots$$



Proof of Lemma 3.10 (cont'd)

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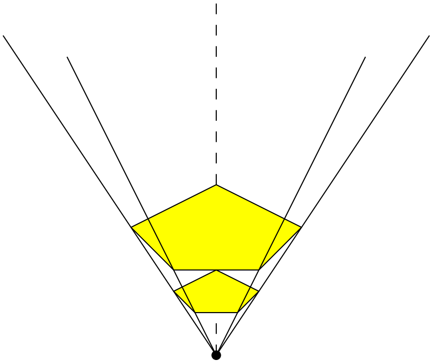

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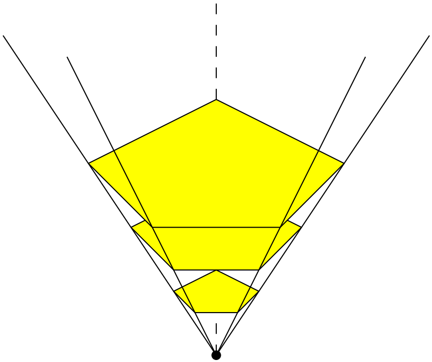

Proof of Lemma 3.10 (cont'd)

$$\sigma_{\mathrm{cone}(\mathcal{P})}(1,1,\ldots,1,z_{d+1}) = 1 + \sum_{t\geq 1} \sigma_{t\mathcal{P}}(1,1,\ldots,1)\, z_{d+1}^{t}$$

Proof of Lemma 3.10 (cont'd)

Since $\sigma_{\mathcal{P}}(1, 1, \ldots, 1) = \#\left(\mathcal{P} \cap \mathbb{Z}^d\right)$,

$$\begin{aligned}\sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \ldots, 1, z_{d+1}) &= 1 + \sum_{t \geq 1} \sigma_{t\mathcal{P}}(1, 1, \ldots, 1)\, z_{d+1}^t \\ &= 1 + \sum_{t \geq 1} \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right) z_{d+1}^t\end{aligned}$$

Proof of Lemma 3.10 (cont'd)

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Back to Proof of Theorem 3.8

- Reminder: Enough to show for an integral d-simplex Δ

$$\mathrm{Ehr}_{\Delta}(z) = 1 + \sum_{t \geq 1} L_{\Delta}(t)\, z^t = \frac{g(z)}{(1-z)^{d+1}}$$

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Back to Proof of Theorem 3.8

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- $\mathrm{Ehr}_{\Delta}(z) = \sigma_{\mathrm{cone}(\Delta)}(1, 1, \ldots, 1, z)$ (Lem 3.10)

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- Denote the $d+1$ vertices of Δ by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$

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- Let's look at $\sigma_{\mathrm{cone}(\Delta)}(z_1, z_2, \ldots, z_{d+1})$
- $\mathrm{cone}(\Delta) \subset \mathbb{R}^{d+1}$ is simplicial, with apex the origin and generators

$$\mathbf{w}_1 = (\mathbf{v}_1, 1),\ \mathbf{w}_2 = (\mathbf{v}_2, 1),\ \ldots,\ \mathbf{w}_{d+1} = (\mathbf{v}_{d+1}, 1) \in \mathbb{Z}^{d+1}$$

Proof of Theorem 3.8 (cont'd)

- Then

$$\sigma_{\mathrm{cone}(\Delta)}\left(z_1, \ldots, z_{d+1}\right) = \frac{\sigma_{\Pi}\left(z_1, \ldots, z_{d+1}\right)}{\left(1 - \mathbf{z}^{\mathbf{w}_1}\right) \cdots \left(1 - \mathbf{z}^{\mathbf{w}_{d+1}}\right)},$$

where $\Pi = \{\lambda_1 \mathbf{w}_1 + \cdots + \lambda_{d+1} \mathbf{w}_{d+1} : 0 \leq \lambda_1, \ldots, \lambda_{d+1} < 1\}$

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$$\sigma_{\mathrm{cone}(\Delta)}(z_1, \ldots, z_{d+1}) = \frac{\sigma_{\Pi}(z_1, \ldots, z_{d+1})}{(1 - \mathbf{z}^{\mathbf{w}_1}) \cdots (1 - \mathbf{z}^{\mathbf{w}_{d+1}})},$$

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- Note: σ_{Π} is a Laurent polynomial in $z_1, z_2, \ldots, z_{d+1}$

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- Note: σ_Π is a Laurent polynomial in $z_1, z_2, \ldots, z_{d+1}$
- Claim: The z_{d+1}-degree of σ_Π is at most d

Proof of Theorem 3.8: Proof of Claim

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$$\sigma_{\Pi}(1, \ldots, 1, 1) = \sum_{\mathbf{m} \in \Pi \cap \mathbb{Z}^{d+1}} \mathbf{1}^{\mathbf{m}}$$

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- This finishes the proof $\square$