Discrete Mathematics & Computational Structures
Lattice-Point Counting in Convex Polytopes
(2) A garelly of discrete volumes

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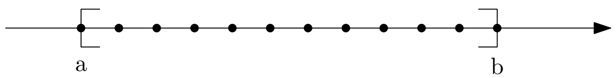
Convex polytopes in dimension 1

Convex polytopes in dimension 1

= straight line segments

- Integral segment $[a, b]$, $a, b \in \mathbb{Z}, a < b$

$$\#([a, b] \cap \mathbb{Z}) = b - a + 1$$



- Rational segment $[a/b, c/d]$, $a, b, c, d \in \mathbb{Z}, a/b < c/d$

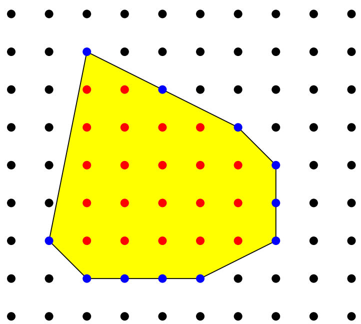
$$\#([a/b, c/d] \cap \mathbb{Z}) = \lfloor c/d \rfloor - \lfloor (a-1)/b \rfloor$$

(Exercise 2.1)

Convex polytopes in dimension 2

Convex polytopes in dimension 2

= convex polygons



Lattice-point counting is a topic of Sect. 2.6 and 2.7

Convex polytopes in dimension d

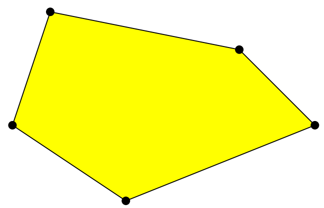
Convex polytopes in dimension d

= convex hulls of a finite set of points

For $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset \mathbb{R}^d$

$$\mathcal{P} = \left\{ \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n : \begin{array}{l} \text{all } \lambda_k \geq 0 \text{ and} \\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1 \end{array} \right\},$$

that is, the smallest convex set containing $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$



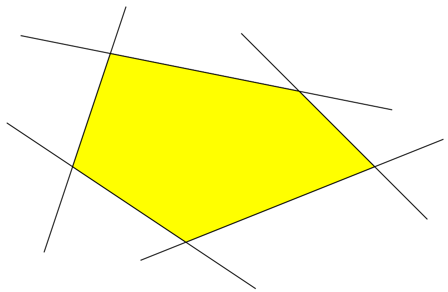
Convex polytopes in dimension d, another point of view

Convex polytopes in dimension d

= bounded intersections of finitely many half-spaces

For $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m \in \mathbb{R}^d$ and $b_1, b_2, \dots, b_m \in \mathbb{R}$

$$\mathcal{P} = \{\mathbf{x} : \mathbf{a}_i \cdot \mathbf{x} \leq b_i \text{ for all } i = 1, 2, \dots, m\}$$



Convex polytopes

Convex polytopes are

- convex hulls of a finite set of points (vertex description, v-polytopes)

or

- bounded intersections of finitely many half-spaces (hyperplane description, h-polytopes)

Theorem (Main theorem for polytopes, Minkowski-Weyl Theorem)

- Every v-polytope is an h-polytope
- Every h-polytope is a v-polytope

Proof: See Appendix A in the textbook

Dimension of convex polytopes

$\mathcal{P}$ a convex polytope

Definition (Dimension)

The dimension of $\mathcal{P}$ is the dimension of the span of $\mathcal{P}$, where

$$\operatorname{span} \mathcal{P} := \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}\}$$

Definition (d-Polytope)

If the dimension of $\mathcal{P}$ is d, then we write

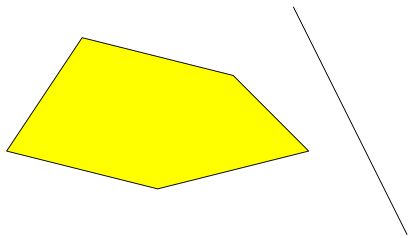
- $\dim \mathcal{P} = d$
- $\mathcal{P}$ is a d-polytope

Valid inequalities

$\mathcal{P} \subseteq \mathbb{R}^d$ a convex polytope; $\mathbf{a} \in \mathbb{R}^d$, $b \in \mathbb{R}$

Definition (Valid inequality)

The inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ is a valid inequality for $\mathcal{P}$ if $\mathbf{a} \cdot \mathbf{z} \leq b$ for all $\mathbf{z} \in \mathcal{P}$



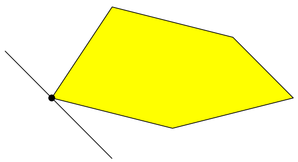
Faces of a convex polytope

$\mathcal{P} \subseteq \mathbb{R}^d$ a convex polytope

Definition (Face)

$\mathcal{F}$ is a face of $\mathcal{P}$ if $\exists$ a valid inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ for $\mathcal{P}$ s.t.

$$\mathcal{F} = \mathcal{P} \cap \{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = b\}$$



Remark

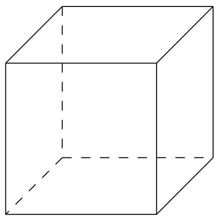
- Every face of a convex polytope is also a convex polytope
- $\mathcal{P}$ and $\varnothing$ are faces of $\mathcal{P}$

Names of faces

$\mathcal{P} \subseteq \mathbb{R}^d$ a convex polytope

Definition

0-dimensional face	vertex of $\mathcal{P}$
1-dimensional face	edge of $\mathcal{P}$
$(d-1)$-dimensional face	facet of $\mathcal{P}$



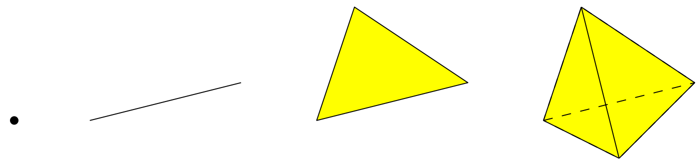
Simplices

Fact

Every convex d-polytope has at least $d+1$ vertices

Definition (Simplex)

A convex d-polytope is a d-simplex if it has exactly $d+1$ vertices



Integral polytopes and rational polytopes

Definition (Integral polytope)

A convex polytope is integral if all of its vertices have integer coordinates

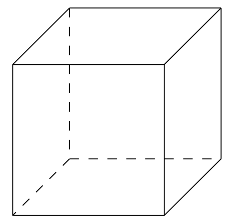
Definition (Rational polytope)

A convex polytope is rational if all of its vertices have rational coordinates

Unit cubes

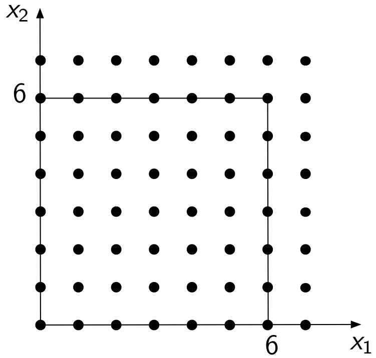
Definition (Unit d-cube $\square$)

- Vertices
$$\{(x_1, x_2, \ldots, x_d) : \text{ all } x_k = 0 \text{ or } 1\}$$
- Hyperplane description
$$\square = \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 \le x_k \le 1 \text{ for all } k\right\}.$$



Question

What's the number of integer points in $t\square$ ($t \in \mathbb{Z}_{>0}$)?



$$\#\left(t\square \cap \mathbb{Z}^d\right) = \#\left([0,t]^d \cap \mathbb{Z}^d\right) = (t+1)^d$$

Lattice-point enumerators

$\mathcal{P} \subseteq \mathbb{R}^d$ not necessarily a convex polytope

Definition (Lattice-point enumerator)

The lattice-point enumerator fot $t\mathcal{P}$ is defined as

$$L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$$

Remarks

- $L_{\mathcal{P}}(t) = \#\left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right)$
- $L_{\square}(t) = (t+1)^d$

$(t+1)^d$ is the generating function of ...

$$(t+1)^d = \sum_{k=0}^{d} \binom{d}{k} t^k,$$

where $\binom{d}{k}$ is a binomial coefficient defined as follows

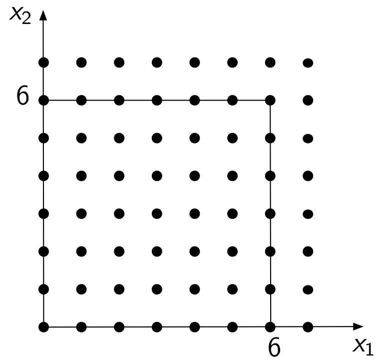
Definition (Binomial coefficient)

For $m \in \mathbb{C}, n \in \mathbb{Z}_{>0}$

$$\binom{m}{n} := \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} \tag{1}$$

What about the interior?

$$L_{\square^\circ}(t) = \#\left(t\square^\circ \cap \mathbb{Z}^d\right) = \#\left((0,t)^d \cap \mathbb{Z}^d\right) = (t-1)^d$$



Remark

$L_{\square^\circ}(t) = (-1)^d L_{\square}(-t)$

Ehrhart series — another tool for studying discrete volume

$\mathcal{P} \subseteq \mathbb{R}^d$

Definition (Ehrhart series)

The Ehrhart series of $\mathcal{P}$ is the generating fn of $L_\mathcal{P}(t)$:

$$\mathrm{Ehr}_\mathcal{P}(z) := 1 + \sum_{t \geq 1} L_\mathcal{P}(t)\, z^t$$

Namely,

$$\mathrm{Ehr}_\square(z) = 1 + \sum_{t \geq 1} (t+1)^d\, z^t$$

Eulerian numbers

Definition (Eulerian number)

For $0 \leq k \leq d$, the Eulerian number $A(d,k)$ is defined through

$$\sum_{j \geq 0} j^d\, z^j = \frac{\sum_{k=0}^d A(d,k)\, z^k}{(1-z)^{d+1}} \qquad (2)$$

Then,

$$\mathrm{Ehr}_\square(z) = 1 + \sum_{t \geq 1} (t+1)^d\, z^t = \sum_{t \geq 0} (t+1)^d\, z^t = \frac{1}{z} \sum_{t \geq 1} t^d\, z^t$$
$$= \frac{\sum_{k=1}^d A(d,k)\, z^{k-1}}{(1-z)^{d+1}}$$

What's an Eulerian number?

Fact

$A(d,k) = \#$ permutations of $\{1, \ldots, d\}$ with $k-1$ ascents

$d = 6$, $k = 3$: 1 4 | 2 5 6 | 3

Properties (Exercise 2.8) $1 \leq k \leq d$

$$A(d,k) = A(d, d+1-k),$$
$$A(d,k) = (d-k+1)\, A(d-1, k-1) + k\, A(d-1,k),$$
$$\sum_{k=0}^d A(d,k) = d!, \qquad (3)$$
$$A(d,k) = \sum_{j=0}^k (-1)^j \binom{d+1}{j} (k-j)^d$$

Summary: The unit cube

Theorem 2.1

(a) The lattice-point enumerator of $\square$ is the polynomial

$$L_\square(t) = (t+1)^d = \sum_{k=0}^d \binom{d}{k} t^k$$

(b) Its evaluation at negative integers yields the relation

$$(-1)^d L_\square(-t) = L_{\square^\circ}(t)$$

(c) The Ehrhart series of $\square$ is $\mathrm{Ehr}_\square(z) = \dfrac{\sum_{k=1}^d A(d,k)\, z^{k-1}}{(1-z)^{d+1}}$ □

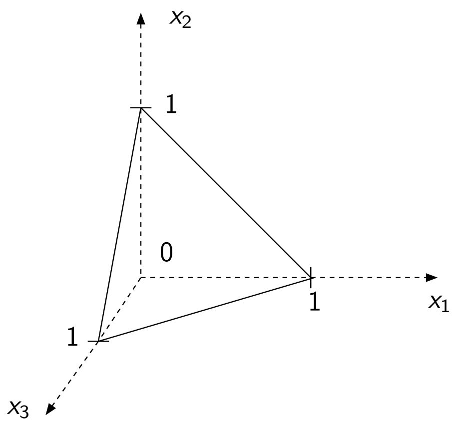
Standard simplices

Definition (Standard d-simplex)

- Vertices
  $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ and $\mathbf{0}$, where
  $\mathbf{e}_j$ is the unit vector $(0, \ldots, 1, \ldots, 0)$ with a 1 in the j-th position
- Hyperplane description

$$\Delta = \left\{ (x_1, x_2 \ldots, x_d) \in \mathbb{R}^d : \begin{array}{l} x_1 + x_2 + \cdots + x_d \leq 1, \\ \text{all } x_k \geq 0 \end{array} \right\}$$

Example: Standard 3-simplex



The dilated standard simplex $t\Delta$

$$t\Delta = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \begin{array}{l} x_1 + x_2 + \cdots + x_d \leq t, \\ \text{all } x_k \geq 0 \end{array} \right\}$$

Let's compute the discrete volume and the Ehrhard series!

We need a trick

- Δ involves an inequality
- The example from Lecture 1 involves equalities only

Trick $\rightarrow$ $\Big\{$ Transform an inequality to equalities by introducing an extra coordinate

Slack variables

- Want to count all integer solutions $(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d_{\geq 0}$ to

$$m_1 + m_2 + \cdots + m_d \leq t \tag{4}$$

- Let $m_{d+1} = \text{RHS} - \text{LHS} \geq 0$
- Then

$$\boxed{\begin{array}{c}\#\text{ integer solutions } (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d_{\geq 0} \text{ to} \\ m_1 + m_2 + \cdots + m_d \leq t\end{array}}$$
$$\|$$
$$\boxed{\begin{array}{c}\#\text{ integer solutions } (m_1, m_2, \ldots, m_{d+1}) \in \mathbb{Z}^{d+1}_{\geq 0} \text{ to} \\ m_1 + m_2 + \cdots + m_{d+1} = t\end{array}}$$

Such a variable m_{d+1} is called a slack variable

Discrete volume of a standard simplex

- Similarly to Lecture 1

$$\begin{aligned}\#\left(t\Delta \cap \mathbb{Z}^d\right) &= \text{const}\left(\left(\sum_{m_1 \geq 0} z^{m_1}\right)\left(\sum_{m_2 \geq 0} z^{m_2}\right)\cdots\left(\sum_{m_{d+1} \geq 0} z^{m_{d+1}}\right) z^{-t}\right) \\ &= \text{const}\left(\frac{1}{(1-z)^{d+1} z^t}\right) \qquad (5)\end{aligned}$$

- Now use the binomial series

$$\frac{1}{(1-z)^{d+1}} = \sum_{k \geq 0} \binom{d+k}{d} z^k \quad \text{for } d \geq 0 \tag{6}$$

- That gives $L_\Delta(t) := \#\left(t\Delta \cap \mathbb{Z}^d\right) = \binom{d+t}{d}$

What about the interior Δ°?

$$\begin{aligned}&L_{\Delta^\circ}(t) \\ &= \#\left\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d_{>0} : m_1 + m_2 + \cdots + m_d < t\right\} \\ &= \#\left\{(m_1, m_2, \ldots, m_{d+1}) \in \mathbb{Z}^{d+1}_{>0} : m_1 + m_2 + \cdots + m_{d+1} = t\right\} \\ &= \text{const}\left(\left(\sum_{m_1 > 0} z^{m_1}\right)\left(\sum_{m_2 > 0} z^{m_2}\right)\cdots\left(\sum_{m_{d+1} > 0} z^{m_{d+1}}\right) z^{-t}\right) \\ &= \text{const}\left(\left(\frac{z}{1-z}\right)^{d+1} z^{-t}\right) \\ &= \text{const}\left(z^{d+1-t}\sum_{k \geq 0}\binom{d+k}{d} z^k\right) \\ &= \binom{t-1}{d} \overset{\text{Ex. 2.10}}{=} (-1)^d \binom{d-t}{d}\end{aligned}$$

Summary: The standard d-simplex

Theorem 2.2

(a) The lattice-point enumerator of Δ is the polynomial

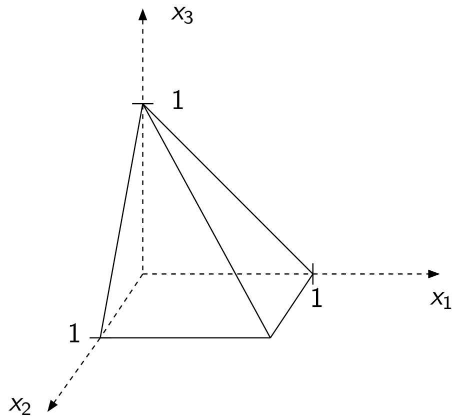
$$L_\Delta(t) = \binom{d+t}{d}$$

(b) Its evaluation at negative integers yields

$$(-1)^d L_\Delta(-t) = L_{\Delta^\circ}(t)$$

(c) The Ehrhart series of Δ is $\text{Ehr}_\Delta(z) = \dfrac{1}{(1-z)^{d+1}}$

Example: The 3-dimensional pyramid



Pyramids

Definition (d-Dimensional pyramid)

- Vertices

$$\{(x_1, x_2, \ldots, x_{d-1}, 0) : \text{ all } x_k = 0 \text{ or } 1\} \cup \{(0, 0, \ldots, 0, 1)\}$$

- Hyperplane description

$$\mathcal{P} = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \begin{array}{l} 0 \le x_k \le 1 - x_d \le 1 \\ \text{for all } k = 1, \ldots, d-1 \end{array} \right\} \tag{9}$$

Remark

d-dim. pyramid $\subseteq$ d-dim. unit cube

Lattice-point enumerator of a pyramid

$$\begin{aligned} L_{\mathcal{P}}(t) &= \left\{ (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : \begin{array}{l} 0 \le m_k \le t - m_d \le t \\ \text{for all } k = 1, \ldots, d-1 \end{array} \right\} \\ &= \sum_{m_d=0}^{t} (t - m_d + 1)^{d-1} \\ &= \sum_{k=1}^{t+1} k^{d-1} \end{aligned}$$

Question

What's the last sum?

Bernoulli polynomials and Bernoulli numbers

Definition (Bernoulli polynomial)

The Bernoulli polynomials $B_k(x)$ are defined via the generating fn

$$\frac{z\, e^{xz}}{e^z - 1} = \sum_{k \geq 0} \frac{B_k(x)}{k!} z^k \tag{8}$$

First few Bernoulli polynomials: $k = 0, 1, 2, 3, \ldots$

$1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}, x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ldots$

Definition (Bernoulli number)

The Bernoulli numbers are $B_k := B_k(0)$

A lemma to show a connection...

Lemma 2.3

$$\sum_{k=0}^{n-1} k^{d-1} = \frac{1}{d}(B_d(n) - B_d) \quad \text{for integers } d \geq 1,\ n \geq 2$$

Proof:

$$\sum_{d \geq 0} \frac{B_d(n) - B_d}{d!} z^d = z \frac{e^{nz} - 1}{e^z - 1} = z \sum_{k=0}^{n-1} e^{kz}$$

$$= z \sum_{k=0}^{n-1} \sum_{j \geq 0} \frac{(kz)^j}{j!} = \sum_{j \geq 0} \left(\sum_{k=0}^{n-1} k^j \right) \frac{z^{j+1}}{j!}$$

$$= \sum_{j \geq 1} \left(\sum_{k=0}^{n-1} k^{j-1} \right) \frac{z^j}{(j-1)!}$$

and compare the both sides □

Lattice-point enumerator of a pyramid, cont'd

$$L_{\mathcal{P}}(t) = \left\{ (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : \begin{array}{l} 0 \leq m_k \leq 1 - m_d \leq 1 \\ \text{for all } k = 1, \ldots, d{-}1 \end{array} \right\}$$

$$= \sum_{m_d=0}^{t} (t - m_d + 1)^{d-1}$$

$$= \sum_{k=1}^{t+1} k^{d-1}$$

$$= \sum_{k=0}^{t+1} k^{d-1} \quad (\text{if } d \geq 2)$$

$$= \frac{1}{d}(B_d(t+2) - B_d) \tag{10}$$

How about the interior $\mathcal{P}^\circ$?

$$L_{\mathcal{P}^\circ}(t) = \left\{ (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : \begin{array}{l} 0 < m_k < 1 - m_d < 1 \\ \text{for all } k = 1, \ldots, d{-}1 \end{array} \right\}$$

$$= \sum_{m_d=1}^{t-1} (t - m_d - 1)^{d-1}$$

$$= \sum_{k=0}^{t-2} k^{d-1} = \frac{1}{d}(B_d(t-1) - B_d)$$

From Exercises 2.15 and 2.16

$$L_{\mathcal{P}}(-t) = \frac{1}{d}(B_d(-t+2) - B_d) = (-1)^d \frac{1}{d}(B_d(t-1) - B_d)$$
$$= (-1)^d L_{\mathcal{P}^\circ}(t)$$

Pyramids over polytopes — generalization

$\mathcal{Q}$ a convex $(d-1)$-polytope, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ the vertices of $\mathcal{Q}$

Definition (Pyramid over a polytope)

The pyramid over $\mathcal{Q}$ is the convex hull of $(\mathbf{v}_1, 0)$, $(\mathbf{v}_2, 0)$, $\dots$, $(\mathbf{v}_m, 0)$, and $(0, \dots, 0, 1)$
Denoted by $\mathrm{Pyr}(\mathcal{Q})$

Note

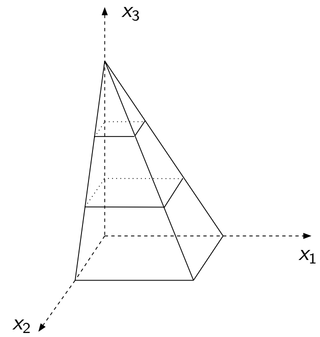
$\mathcal{P} = \mathrm{Pyr}(\square)$

Question

What is $L_{\mathrm{Pyr}(\mathcal{Q})}(t) := \#(t\,\mathrm{Pyr}(\mathcal{Q}) \cap \mathbb{Z}^d)$?

Derivation of $L_{\mathrm{Pyr}(\mathcal{Q})}(t)$

$$L_{\mathrm{Pyr}(\mathcal{Q})}(t) = 1 + L_{\mathcal{Q}}(1) + L_{\mathcal{Q}}(2) + \cdots + L_{\mathcal{Q}}(t) = 1 + \sum_{j=1}^{t} L_{\mathcal{Q}}(j)$$



The Ehrhart series of a pyramid

Theorem 2.4

$$\mathrm{Ehr}_{\mathrm{Pyr}(\mathcal{Q})}(z) = \frac{\mathrm{Ehr}_{\mathcal{Q}}(z)}{1-z}$$

Proof:

$$\begin{aligned}\mathrm{Ehr}_{\mathrm{Pyr}(\mathcal{Q})}(z) &= 1 + \sum_{t\geq 1} L_{\mathrm{Pyr}(\mathcal{Q})}(t)\, z^t = 1 + \sum_{t \geq 1}\left(1 + \sum_{j=1}^{t} L_{\mathcal{Q}}(j)\right) z^t \\ &= \sum_{t\geq 0} z^t + \sum_{t\geq 1}\sum_{j=1}^{t} L_{\mathcal{Q}}(j)\, z^t = \frac{1}{1-z} + \sum_{j\geq 1} L_{\mathcal{Q}}(j) \sum_{t \geq j} z^t \\ &= \frac{1}{1-z} + \sum_{j\geq 1} L_{\mathcal{Q}}(j) \frac{z^j}{1-z} = \frac{1 + \sum_{j\geq 1} L_{\mathcal{Q}}(j)\, z^j}{1-z} \qquad \square\end{aligned}$$

Summary: The pyramid

Theorem 2.5

$\mathcal{P}$ the d-pyramid

$$\mathcal{P} = \left\{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : 0 \leq x_1, x_2, \dots, x_{d-1} \leq 1 - x_d \leq 1\right\}$$

(a) The lattice-point enumerator of $\mathcal{P}$ is the polynomial

$$L_{\mathcal{P}}(t) = \frac{1}{d}\left(B_d(t+2) - B_d\right)$$

(b) Its evaluation at negative integers yields $(-1)^d L_{\mathcal{P}}(-t) = L_{\mathcal{P}^\circ}(t)$

(c) The Ehrhart series of $\mathcal{P}$ is $\mathrm{Ehr}_{\mathcal{P}}(z) = \dfrac{\sum_{k=1}^{d-1} A(d-1,k)\, z^{k-1}}{(1-z)^{d+1}}$

$\square$

Cross-polytopes

Definition (Cross-polytope)

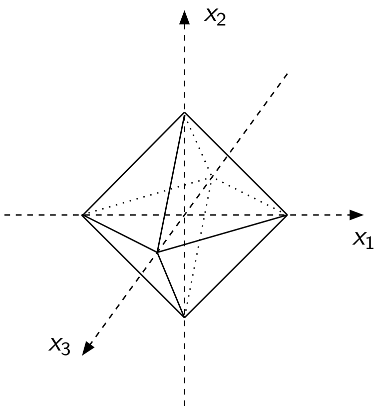
- Vertices

$$\pm\mathbf{e}_1, \pm\mathbf{e}_2, \dots, \pm\mathbf{e}_d$$

- Hyperplane description

$$\lozenge := \left\{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : |x_1| + |x_2| + \cdots + |x_d| \le 1\right\} \tag{14}$$

An octahedron = a 3-dimensional cross-polytope



The bipyramid over a polytope — generalization

$\mathcal{Q}$ a convex $(d-1)$-polytope containing the origin, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ the vertices of $\mathcal{Q}$

Definition (bipyramid over a polytope)

The bipyramid over $\mathcal{Q}$ is the convex hull of $(\mathbf{v}_1, 0)$, $(\mathbf{v}_2, 0), \dots, (\mathbf{v}_m, 0)$, $(0, \dots, 0, 1)$, and $(0, \dots, 0, -1)$
Denoted by $\mathrm{BiPyr}(\mathcal{Q})$

Note

d-dim cross-polytope = BiPyr($(d-1)$-dim cross-polytope)

The lattice-point enumerator and the Ehrhart series of a bipyramid

$$L_{\mathrm{BiPyr}(\mathcal{Q})}(t) = 2 + 2L_{\mathcal{Q}}(1) + 2L_{\mathcal{Q}}(2) + \cdots + 2L_{\mathcal{Q}}(t-1) + L_{\mathcal{Q}}(t)$$
$$= 2 + 2\sum_{j=1}^{t-1} L_{\mathcal{Q}}(j) + L_{\mathcal{Q}}(t)$$

Theorem 2.6

$\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{Q})}(z) = \frac{1+z}{1-z}\,\mathrm{Ehr}_{\mathcal{Q}}(z)$ if $\mathcal{Q}$ contains the origin

Proof: Exercise 2.23

Implication to cross-polytopes

- $\Diamond$ = 0-dim cross-polytope = {origin} $\Rightarrow$

$$\mathrm{Ehr}_{\Diamond}(z) = \frac{1}{1-z}$$

- $\Diamond$ = d-dim cross-polytope $\Rightarrow$

$$\mathrm{Ehr}_{\Diamond}(z) = \frac{(1+z)^d}{(1-z)^{d+1}}$$

Let's derive $L_{\Diamond}(t)$ from $\mathrm{Ehr}_{\Diamond}(z)$!

Lattice-point enumerator of a cross-polytope

$$\mathrm{Ehr}_{\Diamond}(z) = \frac{(1+z)^d}{(1-z)^{d+1}} = \frac{\sum_{k=0}^{d}\binom{d}{k}z^k}{(1-z)^{d+1}} = \sum_{k=0}^{d}\binom{d}{k}z^k\sum_{t\geq 0}\binom{t+d}{d}z^t$$
$$= \sum_{k=0}^{d}\binom{d}{k}\sum_{t\geq k}\binom{t-k+d}{d}z^t$$
$$= \sum_{k=0}^{d}\binom{d}{k}\sum_{t\geq 0}\binom{t-k+d}{d}z^t$$
$$= \sum_{t\geq 0}\sum_{k=0}^{d}\binom{d}{k}\binom{t-k+d}{d}z^t$$

Therefore, for all $t \geq 1$

$$L_{\Diamond}(t) = \sum_{k=0}^{d}\binom{d}{k}\binom{t-k+d}{d}$$

Counting the lattice points in $\Diamond^\circ$

$$L_{\Diamond^\circ}(t) = \#\left\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : |m_1| + |m_2| + \cdots + |m_d| < t\right\}$$
$$= \#\left\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : |m_1| + |m_2| + \cdots + |m_d| \leq t-1\right\}$$
$$= L_{\Diamond}(t-1)$$
$$= \sum_{k=0}^{d}\binom{d}{d-k}\binom{t-1+d-k}{d}$$
$$= (-1)^d\sum_{k=0}^{d}\binom{d}{k}(-1)^d\binom{t-1+k}{d}$$
$$= (-1)^d\sum_{k=0}^{d}\binom{d}{k}\binom{-t-k+d}{d} \quad \text{(by Ex. 2.10)}$$
$$= (-1)^d L_{\Diamond}(-t)$$

Summary: Cross-polytopes

Theorem 2.7

$\Diamond$ the cross-polytope in $\mathbb{R}^d$

(a) The lattice-point enumerator of $\Diamond$ is the polynomial

$$L_\Diamond(t) = \sum_{k=0}^{d} \binom{d}{k} \binom{t-k+d}{d}$$

(b) Its evaluation at negative integers yields $(-1)^d L_\Diamond(-t) = L_{\Diamond^\circ}(t)$

(c) The Ehrhart series of $\mathcal{P}$ is $\mathrm{Ehr}_\Diamond(z) = \frac{(1+z)^d}{(1-z)^{d+1}}$ □

Let's get back to $\mathbb{R}^2$: Pick's theorem

Theme

A strange connection between the number of lattice points and the area of an integral convex polygon

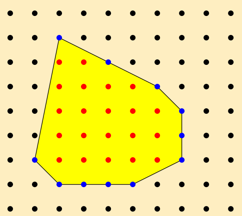
Theorem 2.8 (Pick's theorem)

For an integral convex polygon $\mathcal{P}$

$$A = I + \frac{1}{2}B - 1,$$

where

- A = the area of $\mathcal{P}$
- I = # of lattice points in $\mathcal{P}$
- B = # of lattice points on $\partial\mathcal{P}$

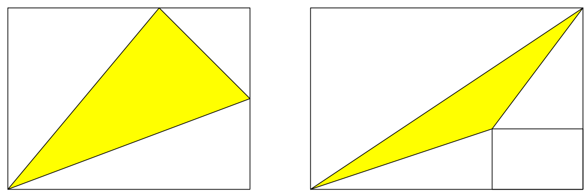


Proof of Pick's theorem (sketch)

- $\mathcal{P}$ is partitioned into $\mathcal{P}_1$ and $\mathcal{P}_2$ ⇒

$$I + \frac{1}{2}B - 1 = (I_1 + \frac{1}{2}B_1 - 1) + (I_2 + \frac{1}{2}B_2 - 1)$$

- ∴ Enough to prove for triangles
- Embed a triangle into a rectangle
- ∴ Enough to prove for right triangles and rectangles
- Ex. 2.24 will finish □



Summary (before the proof): an integral convex polygon

- $\#(\mathcal{P} \cap \mathbb{Z}^2) = I + B = \left(A - \frac{1}{2}B + 1\right) + B = A + \frac{1}{2}B + 1$

Theorem 2.9

(a) The lattice-point enumerator of $\mathcal{P}$ is the polynomial

$$L_{\mathcal{P}}(t) = A\,t^2 + \frac{1}{2}B\,t + 1$$

(b) Its evaluation at negative integers yields the relation

$$L_{\mathcal{P}}(-t) = L_{\mathcal{P}^\circ}(t)$$

(c) The Ehrhart series of $\mathcal{P}$ is

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\left(A - \frac{B}{2} + 1\right) z^2 + \left(A + \frac{B}{2} - 2\right) z + 1}{(1-z)^3}$$

Lattice-point enumerator of an integral convex polygon

Proof of Thm 2.9(a):

- Inflating by factor of t makes
  - the area larger by factor of t^2 (Ex. 2.25)
  - the perimeter larger by factor of t (Ex. 2.25)
- Then, Pick's theorem proves □

Proof of Thm 2.9(b):

$$\begin{aligned} L_{\mathcal{P}^\circ}(t) &= L_{\mathcal{P}}(t) - B\,t \\ &= \left(A\,t^2 + \frac{1}{2}B\,t + 1\right) - B\,t \\ &= A\,t^2 - \frac{1}{2}B\,t + 1 = L_{\mathcal{P}}(-t) \quad \square \end{aligned}$$

Ehrhart series of an integral convex polygon

Proof of Thm 2.9(c):

$$\begin{aligned} \mathrm{Ehr}_{\mathcal{P}}(z) &= 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t \\ &= \sum_{t \geq 0} \left(A\,t^2 + \frac{B}{2}t + 1\right) z^t \\ &= A\,\frac{z^2 + z}{(1-z)^3} + \frac{B}{2}\,\frac{z}{(1-z)^2} + \frac{1}{1-z} \\ &= \frac{\left(A - \frac{B}{2} + 1\right) z^2 + \left(A + \frac{B}{2} - 2\right) z + 1}{(1-z)^3} \quad \square \end{aligned}$$

Roadmap

Contents

d	Integral polytopes		Rational polytopes
general	Sect. 2–5		Sect. 8
	↓		↑
2	Sect. 6	→	Sect. 7

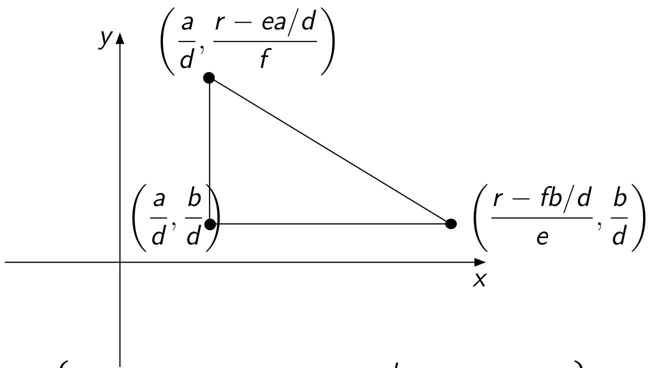
Goal of this section

- Develop a theory for rational convex polygons
- Introduce a quasipolynomial

Steps towards rational convex polygons

- Triangulate a rational convex polygon
- → Enough to study triangles
- Embed a triangle into a rectangle
- → Enough to study right triangles
- Translate, rotate, and mirror a right triangle
- → Enough to study the following type of triangles

A right triangle: setup



$$\mathcal{T} = \left\{ (x, y) \in \mathbb{R}^2 : x \geq \frac{a}{d},\ y \geq \frac{b}{d},\ ex + fy \leq r \right\} \qquad (16)$$

- $a, b, d, e, f, r \in \mathbb{Z}_{\geq 0}$, $ea + fb \leq rd$, $a, b < d$
- For brevity, e, f coprime

Lattice-point enumerator: introducing a slack variable

$$L_{\mathcal{T}}(t) = \#\left\{ (m, n) \in \mathbb{Z}^2 : m \geq \frac{ta}{d},\ n \geq \frac{tb}{d},\ em + fn \leq tr \right\}$$

$$= \#\left\{ (m, n, s) \in \mathbb{Z}^3 : \begin{array}{l} m \geq \frac{ta}{d},\ n \geq \frac{tb}{d},\ s \geq 0, \\ em + fn + s = tr \end{array} \right\}$$

This is interpreted as the coefficient of z^{tr} in the function

$$\left(\sum_{m \geq \frac{ta}{d}} z^{em} \right) \left(\sum_{n \geq \frac{tb}{d}} z^{fn} \right) \left(\sum_{s \geq 0} z^{s} \right),$$

where the subscript under a summation sign means "sum over all integers satisfying this condition"

Lattice-point enumerator: a power series

$$\left(\sum_{m\geq\lceil\frac{ta}{d}\rceil} z^{em}\right)\left(\sum_{n\geq\lceil\frac{tb}{d}\rceil} z^{fn}\right)\left(\sum_{s\geq 0} z^{s}\right) = \frac{z^{\lceil\frac{ta}{d}\rceil e}}{1-z^e}\frac{z^{\lceil\frac{tb}{d}\rceil f}}{1-z^f}\frac{1}{1-z} = \frac{z^{u+v}}{(1-z^e)(1-z^f)(1-z)}, \quad (17)$$

where

$$u := \left\lceil\frac{ta}{d}\right\rceil e \quad \text{and} \quad v := \left\lceil\frac{tb}{d}\right\rceil f \quad (18)$$

Therefore,

$$L_{\mathcal{T}}(t) = \text{const}\left(\frac{z^{u+v-tr}}{(1-z^e)(1-z^f)(1-z)}\right) = \text{const}\left(\frac{1}{(1-z^e)(1-z^f)(1-z)z^{tr-u-v}}\right)$$

Lattice-point enumerator: theorem

$$L_{\mathcal{T}}(t) = \text{const}\left(\frac{1}{(1-z^e)(1-z^f)(1-z)z^{tr-u-v}}\right)$$

- Note: $u+v-tr-e-f-1<0$ (Ex. 2.31)
- A calculation gives the following theorem (Ex. 2.32)

Theorem 2.10

For the triangle $\mathcal{T}$ given by (16), where e and f are coprime,

$$\begin{aligned}L_{\mathcal{T}}(t) &= \frac{1}{2ef}(tr-u-v)^2 + \frac{1}{2}(tr-u-v)\left(\frac{1}{e}+\frac{1}{f}+\frac{1}{ef}\right)\\ &+ \frac{1}{4}\left(1+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f}+\frac{f}{e}+\frac{1}{ef}\right)\\ &+ \frac{1}{e}\sum_{j=1}^{e-1}\frac{\xi_e^{j(v-tr)}}{\left(1-\xi_e^{jf}\right)\left(1-\xi_e^{j}\right)} + \frac{1}{f}\sum_{l=1}^{f-1}\frac{\xi_f^{l(u-tr)}}{\left(1-\xi_f^{le}\right)\left(1-\xi_f^{l}\right)}\end{aligned}$$

□

Properties of this $L_{\mathcal{T}}(t)$

$$\begin{aligned}L_{\mathcal{T}}(t) &= \frac{1}{2ef}(tr-u-v)^2 + \frac{1}{2}(tr-u-v)\left(\frac{1}{e}+\frac{1}{f}+\frac{1}{ef}\right)\\ &+ \frac{1}{4}\left(1+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f}+\frac{f}{e}+\frac{1}{ef}\right)\\ &+ \frac{1}{e}\sum_{j=1}^{e-1}\frac{\xi_e^{j(v-tr)}}{\left(1-\xi_e^{jf}\right)\left(1-\xi_e^{j}\right)} + \frac{1}{f}\sum_{l=1}^{f-1}\frac{\xi_f^{l(u-tr)}}{\left(1-\xi_f^{le}\right)\left(1-\xi_f^{l}\right)}\end{aligned}$$

- $L_{\mathcal{T}}(t)$ is a quadratic fn if we forget the last two sums and u, v
- the last two sums are periodic
- $u = \lceil\frac{ta}{d}\rceil e$ and $v = \lceil\frac{tb}{d}\rceil f$ show periodic behaviors

Therefore, $L_{\mathcal{T}}(t)$ is a "quadratic polynomial" in t whose coefficients are periodic in t

Quasipolynomials

Definition (Quasipolynomial)

A function Q in t is quasipolynomial if Q can be expressed as

$$Q(t) = c_n(t)\ t^n + \cdots + c_1(t)\ t + c_0(t),$$

where $c_0, \ldots, c_n$ are periodic functions in t

- The degree of Q is n (assuming that c_n is not the zero function)
- The period of Q is the least common period of $c_0, \ldots, c_n$

Constituents of a quasipolynomial

Q a quasipolynomial in t

- $\exists$ k and polynomials $p_0, p_1, \ldots, p_{k-1}$ s.t.

$$Q(t) = \begin{cases} p_0(t) & \text{if } t \equiv 0 \bmod k, \\ p_1(t) & \text{if } t \equiv 1 \bmod k, \\ \vdots & \\ p_{k-1}(t) & \text{if } t \equiv k-1 \bmod k \end{cases}$$

- The minimal such k is the period of Q

Definition (Constituent)

For this minimal k, the polynomials $p_0, p_1, \ldots, p_{k-1}$ are the constituents of Q

The lattice-point enumerator of a rational polygon is a quasipolynomial

$$\begin{aligned} L_{\mathcal{T}}(t) &= \frac{1}{2ef}(tr-u-v)^2 + \frac{1}{2}(tr-u-v)\left(\frac{1}{e}+\frac{1}{f}+\frac{1}{ef}\right) \\ &+ \frac{1}{4}\left(1+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f}+\frac{f}{e}+\frac{1}{ef}\right) \\ &+ \frac{1}{e}\sum_{j=1}^{e-1}\frac{\xi_e^{j(v-tr)}}{\left(1-\xi_e^{jf}\right)\left(1-\xi_e^{j}\right)} + \frac{1}{f}\sum_{l=1}^{f-1}\frac{\xi_f^{l(u-tr)}}{\left(1-\xi_f^{le}\right)\left(1-\xi_f^{l}\right)} \end{aligned}$$

- This is a quasipolynomial of degree 2

Theorem 2.11

$\mathcal{P}$ any *rational polygon* $\Rightarrow$

- $L_{\mathcal{P}}(t)$ is a quasipolynomial of degree 2
- Its leading coefficient is the area of $\mathcal{P}$ (in particular, a constant)

Roadmap

Contents

d	Integral polytopes		Rational polytopes
general	Sect. 2–5		Sect. 8
	↓		↑
2	Sect. 6	→	Sect. 7

Goal of this section

- Develop a theory for rational convex polytopes

Rational polytopes: setup

Definition (recap)

A polytope $\mathcal{P}$ is rational if all of its vertices have rational coordinates

We are interested in $\#(t\,\mathcal{P} \cap \mathbb{Z}^d)$

- Consider a hyperplane description of $\mathcal{P}$
- Every coefficient can be chosen as an integer (Ex. 2.7)
- Inequalities are transformed into equalities (by slack var's)
- All pts in $\mathcal{P}$ have nonnegative coord's (by translation)

Therefore, any rational polytope $\mathcal{P}$ is expressed as

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = \mathbf{b}\right\} \tag{23}$$

for some integral matrix $\mathbf{A} \in \mathbb{Z}^{m \times d}$ and some integer vector $\mathbf{b} \in \mathbb{Z}^m$

Note: d is not necessarily the dimension of $\mathcal{P}$

Lattice-point enumerator of a rational polytope

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = \mathbf{b}\right\}$$

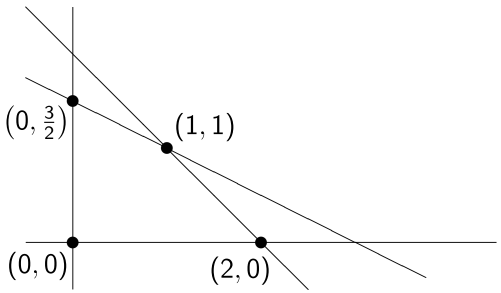
Therefore,

$$\begin{aligned} t\mathcal{P} &= \left\{t\,\mathbf{x} \in \mathbb{R}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = \mathbf{b}\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : \mathbf{A}\,\frac{\mathbf{x}}{t} = \mathbf{b}\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = t\mathbf{b}\right\} \end{aligned}$$

Namely,

$$L_{\mathcal{P}}(t) = \#\left\{\mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = t\mathbf{b}\right\} \tag{24}$$

Example: Setup



$$\mathcal{P} = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, \begin{array}{r} x_1 + 2x_2 \leq 3, \\ x_1 + x_2 \leq 2 \end{array}\right\}$$

Example: Lattice-point enumerator

$$\begin{aligned} L_{\mathcal{P}}(t) &= \#\left\{(x_1, x_2) \in \mathbb{Z}^2 : x_1, x_2 \geq 0, \begin{array}{r} x_1 + 2x_2 \leq 3t, \\ x_1 + x_2 \leq 2t \end{array}\right\} \\ &= \#\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : \begin{array}{r} x_1, x_2, x_3, x_4 \geq 0, \\ x_1 + 2x_2 + x_3 = 3t, \\ x_1 + x_2 + x_4 = 2t \end{array}\right\} \\ &= \#\left\{\mathbf{x} \in \mathbb{Z}^4_{\geq 0} : \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3t \\ 2t \end{pmatrix}\right\} \end{aligned}$$

Example: Power series

$$f(z_1, z_2) := \frac{1}{(1 - z_1 z_2)(1 - z_1^2 z_2)(1 - z_1)(1 - z_2)\, z_1^{3t} z_2^{2t}}$$
$$= \left(\sum_{n_1 \geq 0} (z_1 z_2)^{n_1}\right)\left(\sum_{n_2 \geq 0} (z_1^2 z_2)^{n_2}\right)\left(\sum_{n_3 \geq 0} z_1^{n_3}\right)\left(\sum_{n_4 \geq 0} z_2^{n_4}\right)\frac{1}{z_1^{3t} z_2^{2t}}$$
$$= \sum_{n_1, \ldots, n_4 \geq 0} z_1^{n_1 + 2n_2 + n_3 - 3t} z_2^{n_1 + n_2 + n_4 - 2t}$$

Therefore,

$$L_{\mathcal{P}}(t) = \operatorname*{const}_{z_1, z_2} f(z_1, z_2)$$

Then, we have (Ex. 2.36)

$$\frac{7}{4} t^2 + \frac{5}{2} t + \frac{7 + (-1)^t}{8}$$

General case: Lattice-point enumerator and power series

Reminder

$L_{\mathcal{P}}(t) = \#\left\{\mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = t\mathbf{b}\right\}$

Let

- $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_d$ the columns of $\mathbf{A}$
- $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

Let

$$f(\mathbf{z}) = \frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1})(1 - \mathbf{z}^{\mathbf{c}_2}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{t\mathbf{b}}} \tag{25}$$
$$= \left(\sum_{n_1 \geq 0} \mathbf{z}^{n_1 \mathbf{c}_1}\right)\left(\sum_{n_2 \geq 0} \mathbf{z}^{n_2 \mathbf{c}_2}\right) \cdots \left(\sum_{n_d \geq 0} \mathbf{z}^{n_d \mathbf{c}_d}\right)\frac{1}{\mathbf{z}^{t\mathbf{b}}},$$

where $\mathbf{z}^{\mathbf{c}} := z_1^{c_1} z_2^{c_2} \cdots z_m^{c_m}$ for $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{Z}^m$

General case: A typical term

$$f(\mathbf{z}) = \left(\sum_{n_1 \geq 0} \mathbf{z}^{n_1 \mathbf{c}_1}\right)\left(\sum_{n_2 \geq 0} \mathbf{z}^{n_2 \mathbf{c}_2}\right) \cdots \left(\sum_{n_d \geq 0} \mathbf{z}^{n_d \mathbf{c}_d}\right)\frac{1}{\mathbf{z}^{t\mathbf{b}}}$$

- The exponent of a typical term looks like
$$n_1 \mathbf{c}_1 + n_2 \mathbf{c}_2 + \cdots + n_d \mathbf{c}_d - t\mathbf{b} = \mathbf{A}\mathbf{n} - t\mathbf{b}\,,$$
where $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d_{\geq 0}$
- Therefore, the constant term of $f(\mathbf{z})$ counts the number of solutions $\mathbf{n}$ to
$$\mathbf{A}\mathbf{n} - t\mathbf{b} = \mathbf{0},$$
namely, the number of lattice points in $t\mathcal{P}$

General case: Euler's generating function

Theorem 2.13

Suppose the rational polytope $\mathcal{P}$ is given by (23).
Then the lattice-point enumerator of $\mathcal{P}$ can be computed as

$$L_{\mathcal{P}}(t) = \operatorname*{const}_{\mathbf{z}} \left(\frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1})(1 - \mathbf{z}^{\mathbf{c}_2}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{t\mathbf{b}}}\right) \quad \square$$

General case: Ehrhart series

Corollary 2.14

Suppose the rational polytope $\mathcal{P}$ is given by (23).
Then the Ehrhart series of $\mathcal{P}$ can be computed as

$$\mathrm{Ehr}_{\mathcal{P}}(x) = \operatorname*{const}_{\mathbf{z}} \left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\left(1-\frac{x}{\mathbf{z}^{\mathbf{b}}}\right)} \right)$$

Proof:

$$\mathrm{Ehr}_{\mathcal{P}}(x) = \sum_{t \geq 0} \operatorname*{const}_{\mathbf{z}} \left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{t\mathbf{b}}} \right) x^t$$

$$= \operatorname*{const}_{\mathbf{z}} \left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})} \sum_{t \geq 0} \frac{x^t}{\mathbf{z}^{t\mathbf{b}}} \right)$$

$$= \operatorname*{const}_{\mathbf{z}} \left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})} \frac{1}{1-\frac{x}{\mathbf{z}^{\mathbf{b}}}} \right) \qquad \square$$

Summary

- Definition of a polytope, and related concepts
- Observation of common phenomena through various examples
  - Lattice-point enumerators are polynomials in t
  - Evaluation at $-t$ gives the lattice-point enumerator of the interior
- Definition of a quasipolynomial
- Lattice-point enumerators of rational polytopes