Overview of today's lecture

- density operator (密度作用素)
- privacy of superdense coding (高密度符号化の秘匿性, 次回)

The density operator is another representation for quantum states of physical objects.
The privacy means that nobody can steal any information from the transmitted qubit of the superdense coding.
Don't you think such proof may be difficult?
The use of density operator seems the easiest way to prove it.

Properties of the trace

$|\varphi_1\rangle, \ldots, |\varphi_n\rangle$: orthonormal basis.

$$\mathrm{Tr}A = \langle\varphi_1|A|\varphi_1\rangle + \cdots + \langle\varphi_n|A|\varphi_n\rangle.\text{(Definition)} \quad (1)$$

The value of the trace does not depend on the choice of ONB $\{|\varphi_1\rangle, \ldots, |\varphi_n\rangle\}$, i.e., two different ONBs give the same value of the trace. See your linear algebra textbook for these facts.

$$\mathrm{Tr}[\alpha A] = \alpha \mathrm{Tr}A.$$

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA]. \quad (2)$$

$$\mathrm{Tr}[A \otimes B] = \mathrm{Tr}A \cdot \mathrm{Tr}B. \quad (3)$$

Density operator 1

Suppose that we do not completely know the state of a system, and that we know the state is $|\varphi_i\rangle$ with probability p_i.

Suppose also that we measure an observable

$$A = \sum_{k=1}^{n} k|\psi_k\rangle\langle\psi_k|,$$

where all eigenspaces are of dimension 1.

The probability of getting the measurement outcome k is

$$\begin{aligned}
& \sum_{i=1}^{n} p_i \||\psi_k\rangle\langle\psi_k||\varphi_i\rangle\|^2 \\
= \; & \sum_{i=1}^{n} p_i \langle\varphi_i||\psi_k\rangle\langle\psi_k||\psi_k\rangle\langle\psi_k||\varphi_i\rangle \\
= \; & \sum_{i=1}^{n} p_i \langle\varphi_i||\psi_k\rangle\langle\psi_k||\varphi_i\rangle \\
= \; & \sum_{i=1}^{n} p_i \mathrm{Tr}[|\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k|] \qquad (4) \\
= \; & \mathrm{Tr}\left[\left(\sum_{i=1}^{n} p_i|\varphi_i\rangle\langle\varphi_i|\right)|\psi_k\rangle\langle\psi_k|\right]
\end{aligned}$$

Explanation of Eq. (4)

Let $\{|\varphi_i\rangle,\ |u_2\rangle,\ \ldots,\ |u_n\rangle\}$ be an ONB. We will use this ONB for computation of the trace below.
Then by the definition of trace

$$
\begin{aligned}
& \mathrm{Tr}[|\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k|] && (5)\\
= \;& \langle\varphi_i||\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k||\varphi_i\rangle + \sum_{j=2}^{n}\langle u_j||\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k||u_j\rangle \\
= \;& \langle\varphi_i||\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k||\varphi_i\rangle && (6)\\
= \;& \langle\varphi_i||\psi_k\rangle\langle\psi_k||\varphi_i\rangle && (7)
\end{aligned}
$$

Density operator 2

The previously presented state of the system can be represented by a density operator

$$\rho = \sum_{i=1}^{n} p_i |\varphi_i\rangle\langle\varphi_i|,$$

and the probability of getting the measurement outcome k is given by $\mathrm{Tr}[\rho|\psi_k\rangle\langle\psi_k|]$.

If we apply an unitary operator U on a system, then the state of system is changed from ρ to $U\rho U^*$.

If the system 1 is in state ρ_1, the system 2 is in state ρ_2, and systems 1 and 2 are NOT entangled with each other, then the state of systems 1 and 2 is $\rho_1 \otimes \rho_2$.

Measurement of subsystem

Suppose that physical systems A and B are in the state

$$\tau_{AB} = \rho_1 \otimes \sigma_1 + \cdots + \rho_n \otimes \sigma_n,$$

where ρ_i's are matrices on the state space A and σ_i's are those on B.
Suppose that we measure an observable of the physical system A

$$M = \sum_{k=1}^{m} k P_k.$$

The probability of getting the measurement outcome k is

$$\begin{aligned}
\mathrm{Tr}[\tau_{AB}(P_k \otimes I)] &= \mathrm{Tr}\left[\sum_{i=1}^{n} (\rho_i \otimes \sigma_i)(P_k \otimes I)\right] && (8)\\
&= \mathrm{Tr}\left[\sum_{i=1}^{n} \rho_i P_k \otimes \sigma_i\right] && (9)\\
&= \sum_{i=1}^{n} \mathrm{Tr}[\rho_i P_k \otimes \sigma_i] && (10)\\
&= \sum_{i=1}^{n} \mathrm{Tr}[\rho_i P_k]\mathrm{Tr}[\sigma_i] && (11)\\
&= \sum_{i=1}^{n} \mathrm{Tr}[\mathrm{Tr}[\sigma_i]\rho_i P_k] && (12)
\end{aligned}$$

$$= \quad \mathrm{Tr}\left[\sum_{i=1}^{n} \left(\mathrm{Tr}[\sigma_i]\rho_i\right) P_k\right] \qquad (13)$$

The probability distribution of measurement outcomes is the same as measuring the state

$$\sum_{i=1}^{n} \mathrm{Tr}[\sigma_i]\rho_i \qquad (14)$$

of the system A. The state (14) is called the partial trace of τ_{AB} over B, and denoted by $\mathrm{Tr}_B[\tau_{AB}]$.

Exercise

Submit your answer to the box in front of Room 311, S3 building, by 17:00 Thursday, if you don't finish by 12:10.

1. Suppose that the system is in the state $|0\rangle$ with probability 0.5 and $|1\rangle$ with probability 0.5. Write the corresponding density operator as a 2×2 matrix.

2. Suppose that the system is in the state $(|0\rangle + |1\rangle)/\sqrt{2}$ with probability 0.5 and $(|0\rangle - |1\rangle)/\sqrt{2}$ with probability 0.5. Write the corresponding density operator as a 2×2 matrix.

3. Let P be an $n \times n$ projection matrix of rank 1 ($n \geq 2$). Show that $\mathrm{Tr}[P] = 1$. (Hint: A projection matrix of rank 1 can be written as $|\varphi\rangle\langle\varphi|$ with some vector $|\varphi\rangle$ with $\||\varphi\rangle\| = 1$.)

4. Let M be a 2×2 Hermitian matrix with its spectral decomposition $M = \lambda_1 P_1 + \lambda_2 P_2$ with $\lambda_1 \neq \lambda_2$. Show that $\mathrm{Tr}P_1 = \mathrm{Tr}P_2 = 1$ by using your answer to Problem 3. (Hint: What are the ranks of P_1 and P_2?)

5. Let $\rho = I/2$, where I is the 2×2 identity matrix. Suppose that the system is in state ρ and we measure the observable M given in Problem 4. Compute the probabilities of getting outcomes λ_1 and λ_2 by using

your answer to Problem 4.

6. Let $|\Psi\rangle = (|0_A 0_B\rangle + |1_A 1_B\rangle)/\sqrt{2}$ be a state of systems A and B. Compute the partial trace of $|\Psi\rangle\langle\Psi|$ over B. $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$ are orthonormal bases of A and B, respectively.

7. Suppose that we measure the observable $M \otimes I$ of the system with state $|\Psi\rangle$. Compute the probabilities of getting outcomes λ_1 and λ_2 by using your answers of Problems 5 and 6. (Hint: What density operator represents the state $|\Psi\rangle$?)

8. If your answers to the previous exercises were evaluated as incorrect, please indicate whether or not you agree to that evaluation. Write which part in today's lecture was difficult for your understanding.