What is quantum information processing?

The research of quantum information explores what can be done under the assumption that all the unitary matrices and the measurements are physically realizable.

We (at least I) do not care how one can implement (realize) a given unitary matrix by a physical device.

The research of quantum computation imposes some restrictions on the set of available unitary matrices because otherwise any computation can be done by a single unitary matrix and we become unable to consider the computational complexity of quantum computation.

Some notations

$$
\begin{aligned}
& |\varphi\rangle \otimes |\psi\rangle \\
= \ & |\varphi\rangle|\psi\rangle \\
= \ & |\varphi\psi\rangle
\end{aligned}
$$

A two-dimensional quantum system is said to be a *qubit*.
Qubit represents QUantum BIT.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Quantum teleportation

- Send a quantum state to a recipient who is spacially apart from the sender.
- The sender DOES NOT send the physical system.
- The sender sends 2 bits information for transmission of 1 qubit.
- The sender and the receiver share the entangled state
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Example: Suppose that the sender is on the earth, and the receiver is in a spaceship far apart from the earth.
A physical object is reproduced at a distant place without sending a physical object. Doesn't it seem like a teleportation??

Controlled NOT

Manipulation of a quantum system is represented by a unitary matrix.
A unitary matrix U can be specified by $U|\varphi\rangle$ for every basis vector $|\varphi\rangle$

U: 4×4 unitary matrix

$$U|00\rangle = |00\rangle, \quad U|01\rangle = |01\rangle,$$

$$U|10\rangle = |11\rangle, \quad U|11\rangle = |10\rangle.$$

The right qubit is negated iff the left qubit is one.
U is similar to the NOT gate on the second qubit controlled by the first qubit.

left qubit: control qubit of CNOT
right qubit: target qubit of CNOT

Teleportation (1)

$|\Psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is shared.
$\alpha|0\rangle + \beta|1\rangle$ is to be sent.

The state of the total system is

$$\begin{aligned} & (\alpha|0\rangle + \beta|1\rangle)|\Psi\rangle \\ = & \frac{1}{\sqrt{2}}[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)] \end{aligned}$$

The sender has the leftmost and the middle qubits, and the receiver has the rightmost qubit.

Teleportation (2)

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

Applying CNOT with
control quibit: leftmost qubit
target qubit: middle qubit

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

Teleportation (3)

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

The matrix H:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \tag{1}$$

Applying H to the leftmost qubit:

$$\begin{aligned} & \frac{1}{2}[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \\ & \quad \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)] \\ = \; & \frac{1}{2}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + \\ & \quad |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)] \end{aligned}$$

Teleportation (4)

$$\frac{1}{2}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)] \quad (2)$$

The sender measures the observable Z_1 of the leftmost qubit, and the Z_2 of the middle qubit, where

$$Z_1 = Z_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Teleportation (5)

The sender sends the measurement outcomes, and the receiver applies the following unitary matrix to the rightmost qubit according to outcomes.

Z_1	Z_2	Receiver's matrix
$+1$	$+1$	2×2 identity matrix
$+1$	-1	X
-1	$+1$	Z
-1	-1	ZX

Then $\alpha|0\rangle + \beta|1\rangle$ is teleported to the receiver (Exercise).

Explanation of teleportation

The sender has this qubit and this qubit. The receiver has this qubit.

Before teleportation:

$$(\alpha|0\rangle + \beta|1\rangle)\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

After teleportation:

$$|??\rangle(\alpha|0\rangle + \beta|1\rangle),$$

where ? is 0 or 1, depending on the measurement outcomes.

Exercise

Submit your answer to the box in front of Room 311, S3 building, by 17:00 Thursday, if you don't finish by 12:10.

1. For each measurement outcome $(\pm 1, \pm 1)$ of (Z_1, Z_2), compute the probability of getting the outcome and the state of three qubit after measurement of the state (2). Answer in the following format: (観測量 (Z_1, Z_2) の各々の測定結果 $(\pm 1, \pm 1)$ について、その測定結果を得る確率と状態 (2) の測定後の状態を計算せよ。以下の形式で回答せよ)

(Z_1, Z_2)	probability	state
$(+1, +1)$	?	?
$(+1, -1)$	?	?
$(-1, +1)$	?	?
$(-1, -1)$	?	?

2. For each measurement outcome, compute the state of three qubits after the receiver applies the matrix to the rightmost qubit. Answer in the following format: (各々の測定結果について、受信者が行列を一番右のキュービットに適用したあとの 3 キュービットの状態を計算せよ。以下の形式で回答せよ。)

(Z_1, Z_2)	state
$(+1, +1)$	?
$(+1, -1)$	?
$(-1, +1)$	?
$(-1, -1)$	?

3. Write all the eigenvalues, an orthonormal basis of every eigenspace, and the spectral decomposition of $X \otimes Z$, where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

You may answer an eigenvector as $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

4. If your answers to the previous exercises were evaluated as incorrect, please indicate whether or not you agree to that evaluation. What topic do you want to be included in this lecture? Do you understand the teleportation? If not, write which part was difficult for your understanding.