Orthonormal basis

$\{|\varphi_1\rangle, \ldots, |\varphi_n\rangle\}$: an orthonormal basis of V.
$|\psi\rangle \in V$ can be written as

$$a_1|\varphi_1\rangle + \cdots + a_n|\varphi_n\rangle.$$

We have

$$\left(\sum_{i=1}^{n} |\varphi_i\rangle\langle\varphi_i|\right)|\psi\rangle = \sum_{i=1}^{n} |\varphi_i\rangle\langle\varphi_i|\psi\rangle = \sum_{i=1}^{n} |\varphi_i\rangle a_i = |\psi\rangle.$$

Thus

$$\sum_{i=1}^{n} |\varphi_i\rangle\langle\varphi_i| = I. \tag{1}$$

Assume $i \neq j$.

$$\begin{aligned} |\varphi_i\rangle\langle\varphi_i||\varphi_j\rangle\langle\varphi_j| &= 0, & (2)\\ |\varphi_i\rangle\langle\varphi_i||\varphi_i\rangle\langle\varphi_i| &= |\varphi_i\rangle\langle\varphi_i|, & (3)\\ (|\varphi_i\rangle\langle\varphi_i|)^* &= (\langle\varphi_i|)^*(|\varphi_i\rangle)^* = |\varphi_i\rangle\langle\varphi_i|. & (4) \end{aligned}$$

Properties of a projector

$$\begin{aligned} P_1 &= |\varphi_1\rangle\langle\varphi_1| + \cdots + |\varphi_m\rangle\langle\varphi_m| \\ P_2 &= |\varphi_{m+1}\rangle\langle\varphi_{m+1}| + \cdots + |\varphi_n\rangle\langle\varphi_n| \end{aligned}$$

$$P_1^* = P_1 \text{ (by Eq. (4))} \tag{5}$$

$$P_1 P_2 = 0 \text{ (by Eq. (2))} \tag{6}$$

$$P_1 P_1 = P_1 \text{ (by Eqs. (2) and (3))} \tag{7}$$

Manipulation of a quantum system

Manipulation of a quantum system *without extracting information* is represented by a unitary matrix U.
A unitary matrix U is a matrix such that $UU^* = I$.

Example:

$$|-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\,|\,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned} XX^* &= I, \\ ZZ^* &= I, \\ X|-\rangle &= |\,|\,\rangle, \\ X|\,|\,\rangle &= |-\rangle, \\ Z|-\rangle &= |-\rangle, \\ Z|\,|\,\rangle &= -|\,|\,\rangle, \end{aligned}$$

Tenser product, or Kronecker product

A: $m \times n$ matrix

B: $p \times q$ matrix

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix}$$

The tensor product of column vectors is defined by regarding column vectors as $m \times 1$ and $p \times 1$ matrices. The tensor product of row vectors is similarly defined.

Properties of tensor products

α: a complex number

$$\begin{aligned}
\alpha(|\varphi\rangle \otimes |\psi\rangle) &= (\alpha|\varphi\rangle) \otimes |\psi\rangle \\
&= |\varphi\rangle \otimes (\alpha|\psi\rangle) \\
(|\varphi_1\rangle + |\varphi_2\rangle) \otimes |\psi\rangle &= |\varphi_1\rangle \otimes |\psi\rangle + |\varphi_2\rangle \otimes |\psi\rangle \\
|\varphi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) &= |\varphi\rangle \otimes |\psi_1\rangle + |\varphi\rangle \otimes |\psi_2\rangle
\end{aligned}$$

(similar relations hold for matrices)

$$\begin{aligned}
(A \otimes B)(|\varphi\rangle \otimes |\psi\rangle) &= (A|\varphi\rangle) \otimes (B|\psi\rangle) \\
&= A|\varphi\rangle \otimes B|\psi\rangle \\
(\langle\varphi_1| \otimes \langle\varphi_2|)(|\psi_1\rangle \otimes |\psi_2\rangle) &= \langle\varphi_1|\psi_1\rangle \cdot \langle\varphi_2|\psi_2\rangle \\
(A \otimes B)^* &= A^* \otimes B^* \\
(A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}
\end{aligned}$$

V, W: linear spaces
$V \otimes W$: linear space **spanned by**
$\{|\varphi\rangle \otimes |\psi\rangle : |\varphi\rangle \in V,\, |\psi\rangle \in W\}$.

$\dim V \otimes W = \dim V \times \dim W$.

Composite system

A quantum system 1 is represented by a linear space $\mathcal{H}_1$.
A quantum system 2 is represented by a linear space $\mathcal{H}_2$.

The quantum system consisting of system 1 and system 2 is represented by a vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Applying a unitary operator U_1 to system 1 is equivalent to applying $U_1 \otimes I$ to the composite system.

Measuring a observable A_1 of system 1 is equivalent to measuring the observable $A_1 \otimes I$ of the composite system.

Entangled state

V, W: linear space
Some vector in $V \otimes W$ cannot be written as $|\varphi\rangle \otimes |\psi\rangle$ for any $|\varphi\rangle \in V$ and $|\psi\rangle \in W$.

$$\frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right\}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \tag{8}$$

$$\begin{pmatrix}a\\b\end{pmatrix}\otimes\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}ac\\ad\\bc\\bd\end{pmatrix} \tag{9}$$

If $ac \neq 0$ and $bd \neq 0$, then $a \neq 0$, $b \neq 0$, $c \neq 0$, and $d \neq 0$. Therefore, Eq. (8) cannot be expressed as Eq. (9).

A quantum state that cannot be expressed as $|\varphi\rangle\otimes|\psi\rangle$ is called an **entangled state**.

Subsystem of a composite system

A composite system consists of systems 1 and 2 is in an entangled state.

$$\Downarrow$$

The state of system 1 cannot be expressed by a state vector.

The state vector is an incomplete representation of quantum states.

$\Downarrow$ But

Any quantum state can always be represented as a state vector of some larger system.

Spectral decomposition

もっと詳しく、例をだす
A, B: Hermitian matrices
Spectral decompositions of A and B:

$$\begin{aligned} A &= \lambda_1 P_1 + \cdots + \lambda_m P_m, \\ B &= \eta_1 Q_1 + \cdots + \eta_n Q_n. \end{aligned}$$

The spectral decomposition of $A \otimes B$ is given by

$$A \otimes B = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \eta_j P_i \otimes Q_j. \tag{10}$$

From the above equation, we can see that the set of eigenvalues of $A \otimes B$ is $\{\lambda_i \eta_j \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$.

Exercises

Submit your answer to the box in front of Room 311, S3 building, by **12:00** Thursday (**May 7th**), if you don't finish them by 12:10.

$$|-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\,|\,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |\Psi\rangle = \frac{|-\rangle \otimes |-\rangle + |\,|\,\rangle \otimes |\,|\,\rangle}{\sqrt{2}}$$

I is the 2×2 identity matrix. When you answer to the following, avoid expanding vectors into their components and use the equalities in p. 3-10 as much as possible. (以下の質問に答えるときベクトルを要素に展開することをなるべく避け、p. 3-10 の関係式をなるべく使え)

1. Show that the length (norm) of $|\Psi\rangle$ is 1.

2. Show that X and Z are unitary matrices.

3. Express $(X \otimes I)|\Psi\rangle$ in terms of $|-\rangle$ and $|\,|\,\rangle$. ($(X \otimes I)|\Psi\rangle$ を $|-\rangle$ と $|\,|\,\rangle$ を用いて表せ.) Hint: Use relations in p. 3-10.

4. Express $(Z \otimes I)|\Psi\rangle$ in terms of $|-\rangle$ and $|\,|\,\rangle$.

5. Suppose that one measures the observable $Z \otimes I$ of the system in the state $|\Psi\rangle$. For each measurement

outcome, calculate the probability of getting the outcome and the state after measurement. (状態 $|\Psi\rangle$ にある系の観測量 $Z \otimes I$ を測定するとする。各々の測定結果について、その結果を得る確率と測定後の状態を計算せよ。)

6. Is $Z \otimes Z$ a Hermitian matrix?

7. Is $Z \otimes Z$ a unitary matrix?

8. Write all the eigenvalues of $Z \otimes Z$ and an orthonormal basis of each eigenspace. After that, compute the spectral decomposition of $Z \otimes Z$.

9. Answer Question 5 with $Z \otimes I$ replaced with $Z \otimes Z$.

10 (Optional). Prove that Eq. (10) is the spectral decomposition of $A \otimes B$. You must calculate the set of eigenvalues of $A \otimes B$ and the projectors onto its eigenspaces. It is not enough to simply prove the equality in Eq. (10).

11 (Optional). Write comments on this lecture. If your answers to the previous exercises were evaluated as incorrect, please indicate whether or not you agree to that evaluation. (授業へのコメントを書いて下さい。もし前回の演習で×になったものがあれば、×になったことに納得できたかどうか書いて下さい。)