There will be NO examination. Your grade will evaluated only by your answers to exercises (and optionally submitted report).

Suggested textbooks on quantum information:
Michael A. Nielsen and Isaac L. Chuang, “Quantum Computation and Quantum Information,” ISBN: 0521635039.
和訳: 量子コンピュータと量子通信〈1〉〜〈3〉,
ISBN: 4274200094, 4274200086, 4274200094

Answers of the previous exercises will be explained.

Minimal explanation of quantum mechanics

Today I will introduce the mathematical model of the quantum theory. It does not include Schrödinger's equation.

Schrödinger's equation is almost always explained in a course on quantum physics. I do not explain that. Schrödinger's equation is required when one wants to know the state of a quantum system as a function of time. We do not have to know it.

State of a quantum system

Quantum system: whatever physical phenomenon. E.g. photon polarization.
The state of a quantum system is represented by a complex vector of length (norm) 1 in a complex linear space.

The dimension of the linear space associated with a quantum system is usually infinite dimensional.
Assumption: The dimension of linear space is always finite in this course.

Notation of vectors

$|\varphi\rangle$: column vector in the quantum physics
$\langle\varphi|$: the complex conjugate of $|\varphi\rangle$.

Example of states of photon polarization

$$|-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$|\,|\,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$|\diagup\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$|\diagdown\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The direction of polarization is represented by that of state vector.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}$ represents the circular polarization, which means that polarization is rotating as the photon moves.

Measurement

Measurement of a quantum system =
an action of extracting information from the system.
A Hermitian matrix represents how to measure a quantum system.

A complex square matrix M is *Hermitian* if $M = M^*$.

Eigenvalue (固有値) and eigenspace (固有空間)

M: complex square matrix
A complex number λ is said to be an *eigenvalue* of M if there exists a nonzero vector $\vec{v}$ such that $M\vec{v} = \lambda\vec{v}$.

Eigenspace belonging to $\lambda =$
$\{\vec{u} \mid M\vec{u} = \lambda\vec{u}\}$.
(固有値 λ に属する固有空間)

Projection onto a subspace

V: linear space
W: subspace of V
$W^{\perp}$: orthogonal complement of W in V

P_W: projection onto W
Any $\vec{v}$ can written uniquely as

$$\vec{v} = \vec{w}_1 + \vec{w}_2$$

with $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^{\perp}$.

$P_W(\vec{v}) = \vec{w}_1$.

How to compute the matrix representation of P_W?
1. Find an orthonormal basis $\{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$ (正規直交基底) of W.
2.
$$P_W = |\psi_1\rangle\langle\psi_1| + \cdots + |\psi_m\rangle\langle\psi_m| \tag{1}$$

Spectral decomposition

M: Hermitian matrix
λ_i: i-th eigenvalue of M ($\lambda_i \neq \lambda_j$)
W_i: eigenspace belonging to λ_i
P_i: projection onto W_i.

$$M = \sum_i \lambda_i P_i$$

The above decomposition is called the spectral decomposition of M.

How to compute spectral decomposition

1. Compute all eigenvalues.
2. For each eigenspace W_i, find an orthonormal basis $\{|\psi_{i1}\rangle, \ldots, |\psi_{im}\rangle\}$ (正規直交基底) of W_i.
3. P_i is given by

$$P_i = \sum_{k=1}^{m} |\psi_{ik}\rangle\langle\psi_{ik}|.$$

Measurement

$\mathcal{H}$: linear space associated with a quantum system
Measurement is described by an observable A, which is a Hermitian matrix on $\mathcal{H}$.
Results of measuring the observable A = eigenvalues of A.

We cannot predict which measurement outcome is obtained before the measurement, e.g. the measurement of polarization. But we can calculate the probability of a measurement outcome.

Probability of getting a measurement outcome

The quantum system is in state $|\varphi\rangle$.
Measuring an observable A.
$\lambda_1, \ldots, \lambda_n$: eigenvalues of A.

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

The probability of getting λ_i as the measurement outcome =

$$\left\| P_i |\varphi\rangle \right\|^2 . \tag{2}$$

α: complex number with $|\alpha| = 1$
Since $|\varphi\rangle$ and $\alpha|\varphi\rangle$ give the same probability distribution of the measurement outcomes, they are physically indistinguishable. $|\varphi\rangle$ and $\alpha|\varphi\rangle$ represent the same quantum state.

Example of an observable

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |\varphi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Since $Z^* = Z$, it is Hermitian.

eigenvalue	eigenvector	projector
$+1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
-1	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

The spectral decomposition of Z:

$$Z = +1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P_1|\varphi\rangle = |\varphi\rangle, P_2|\varphi\rangle = 0.$$

Probability of getting $+1$ as the measurement outcome is 1.
Probability of getting -1 as the measurement outcome is 0.

If polarization is represented as in p. 2-8, Z represents the measurement by the slit $|$ (or $-$).

State after nondestructive measurement

Quantum state is changed by nondestructive measurement.
Measuring an observable A of a system with state $|\varphi\rangle$ nondestructively

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

After getting a measurement outcome λ_i, the state become

$$\frac{P_i|\varphi\rangle}{\|P_i|\varphi\rangle\|}.$$

Example of nondestructive measurement

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |\varphi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, |a|^2 + |b|^2 = 1.$$

$$\frac{P_1|\varphi\rangle}{\|P_1|\varphi\rangle\|} = \begin{pmatrix} a/|a| \\ 0 \end{pmatrix}$$

is equivalent to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which represents the — polarization

$$\frac{P_2|\varphi\rangle}{\|P_2|\varphi\rangle\|} = \begin{pmatrix} 0 \\ b/|b| \end{pmatrix}$$

is equivalent to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which represents the | polarization.

Above equations says that after measuring whether the polarization is — or | , polarization becomes — or | according to the measurement outcome.

Exercises

Submit your answer to the box in front of Room 311, S3 building, by 17:00 Thursday, if you don't finish by 12:10.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, |\varphi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

For Japanese students: 無理に英語で答える必要はありません

1. Is X a Hermitian matrix? (X はエルミート行列か？)

2. Compute the spectral decomposition of X. (X のスペクトル分解を計算せよ)

3. Suppose that one measures the observable X of the system with state $|\varphi\rangle$. For each measurement outcome, compute its probability and the state after getting the outcome. (状態 $|\varphi\rangle$ にある系の観測量 X を測定するとき、各々の測定結果について、その結果を得る確率とその結果を得た後の系の状態を計算せよ)

4. If photon polarization is represented as page 2-8, which polarizations are measured by X? (光子の偏光が2-8ページのように表されているとき、観測量 X によって測られる偏光の向きを答えよ)

The following exercise requires deep understanding of linear algebra taught at the first year of undergraduate. (下記の問は大学１年で教えられる線形代数の深い理解を要求する) You can get the grade evaluation 100 even if you give wrong answers to optional exereecises, but please try to answer them. (optional な問題を間違えても成績評価は100点になり得るがなるべく解答しようとしてみて下さい)

5. (Optional) Prove

$$\sum_{j=1}^{n} \left\| P_j |\varphi\rangle \right\|^2 = 1,$$

where P_i and $|\varphi\rangle$ are as defined in Eq. (2). (P_i と $|\varphi\rangle$ が式 (2) のように定義されるときに上の等式を証明せよ)

6 (Optional). Prove that Eq. (1) is the projection onto W in the sense of page 2-11. Hint: You have to prove that $P_W(\vec{v})$ belongs to W and $\vec{v} - P_W(\vec{v})$ belongs to $W^\perp$ for all $\vec{v} \in V$. (和訳：式 (1) が 2-11 ページで定義した意味での射影になっていることを証明せよ。ヒント：任意の $\vec{v} \in V$ について、$P_W(\vec{v})$ が W に属し、$\vec{v} - P_W(\vec{v})$ が $W^\perp$ に所属することを示す必要がある。

7 (Optional). Write comments on this lecture. If your

answers to the previous exercises were evaluated as incorrect, please indicate whether or not you agree to that evaluation. (授業へのコメントを書いて下さい。もし前回の演習で×になったものがあれば、×になったことに納得できたかどうか書いて下さい。)