Error Correction Codes & Multi-user Communications

Agenda

- Shannon Theory
- History of Error Correction Code
- Linear Block Codes
- Decoding
- Convolution Codes
- Multiple-Access Technique
- Capacity of Multiple Access
- Random Access Methods

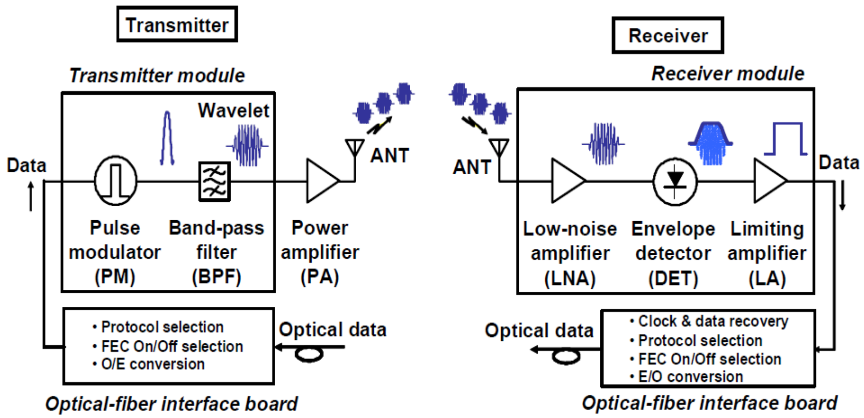
Shannon Theory:

$R < C \rightarrow$ Reliable communication

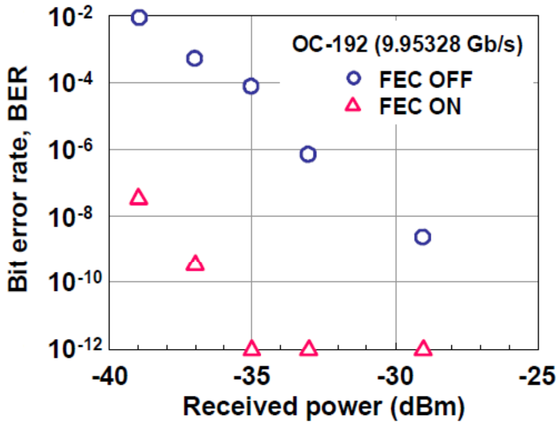
Redundancy (Parity bits)

in transmitted data stream

$\rightarrow$ error correction capability



Reed-Solomon Code(255,247)



Encoding

Block Code	Convolutional Code
Code length is fixed	Coding Rate is fixed

Decoding

Hard-Decoding	Soft-Decoding
Digital Information	Analog Information

History of Error Correction Code

- Shannon (1948): Random Coding
- Golay (1949): Golay Code(Perfect Code)
- Hamming (1950): Hamming Code
  (Single Error Correction, Double Error Detection)
- Gilbert (1952): Gilbert Bound on Coding Rate

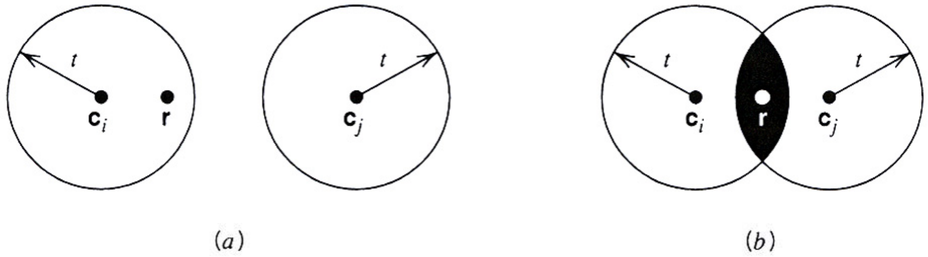
- Muller (1954): Combinatorial Function & ECC
- Elias (1954): Convolutional Code
- Reed ,Solomon (1960): RS Code (Maximal Separable)
- Hocquenghem (1959) ,Bose,Chaudhuri (1960): BCH Code (Multiple Error Correction)
- Peterson (1960): Error Location Polynomial

- Wozencraft , Reiffen (1961): Sequential decoding
- Gallager (1962) :LDPC
- Fano (1963): Fano Decoding Algorithm
- Ziganzirov (1966): Stack Decoding Algorithm
- Forney (1966): Generalized Minimum Distance Decoding (Error and Erasure Decoding)
- Viterbi (1967): Optimal Decoding Algorithm, Dynamic Programming

- Berlekamp (1968): Fast Iterative BCH Decoding
- Forney (1966): Concatinated Code
- Goppa (1970): Rational Function Code
- Justeson (1972): Asymptotically Good Code
- Ungerboeck,Csajka (1976): Trellis Code Modulation,
- Goppa (1980): Algebraic-Geometry Code

- Welch,Berlekamp (1983): Remainder Decoding Algorithm
- Araki, Sorger and Kotter (1993): Fast GMD Decoding Algorithm
- Berrou (1993): Turbo Code(Parallel concatinated convolutional code)

Basics of Decoding



a) Hamming distance $d(\mathbf{c}_i, \mathbf{c}_j) \geq 2t+1$

b) Hamming distance $d(\mathbf{c}_i, \mathbf{c}_j) < 2t$
The received vector is denoted by **r**.
$\boldsymbol{t} \rightarrow$ **errors correctable**

Linear Block Codes

$(n, k, d_{\min})$ code

n : code length

k : number of information bits

$d_{\min}$: minimum distance

k/n : coding rate

For large d ,Good Error correction capability
R=k / n (Low coding rate)

Trade-off between error correction and coding rate

(n, k, d) Linear Block Code is
Linear Subspace with k-dimension in n-dimension linear space.

Arithmetic operations $(+, -, \times, /)$ for encoding and decoding over an **finite field** GF(Q)

where $Q = p^r$, p: prime number r: positive integer

Example GF(2):

addition (XOR)

+	0	1
0	0	1
1	1	0

multiplication (AND)

·	0	1
0	0	0
1	0	1

Analog and Digital Arithmetic Operation

- Analog: Real Number Field [R],
  Complex Number Field [C]
- Digital: Finite Field [GF(Q)]

[Encoder]

- The Generator Matrix **G** and the Parity Check Matrix **H**

k information bits **X** → encoder **G** → n-bits codeword **C**

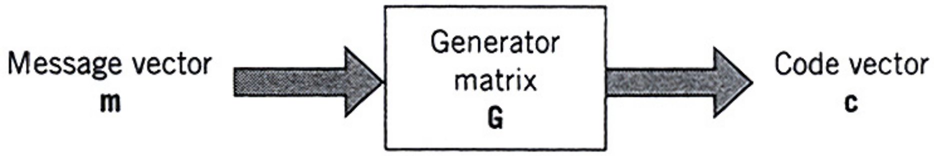
$$\mathbf{C} = \mathbf{X}\mathbf{G}$$

- Dual (n, n - k) code

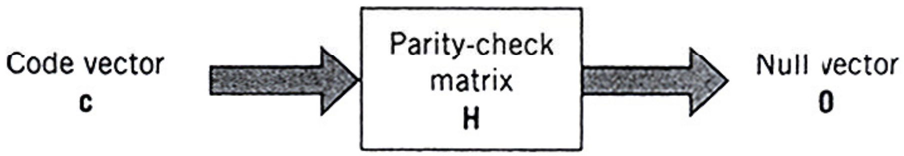
Complement orthogonal subspace
Parity Check Matrix **H** = Generator Matrix of Dual code

$$\mathbf{C}\mathbf{H}^t = 0$$

$$\mathbf{G}\mathbf{H}^t = 0$$



(a)



(b)

error vector & syndrome

$\mathbf{c}$: codeword vector

$\mathbf{e}$: error vector

$\mathbf{r}$: received vector (after Hard decision)

$\mathbf{s}$: syndrome

$$\mathbf{s} = \mathbf{r}H^t = (\mathbf{c} + \mathbf{e})H^t = \mathbf{e}H^t$$

$$\mathbf{s} \rightarrow \mathbf{e} \text{ (decoding process)}$$

[Minimum Distance]

Singleton Bound

If no more than $d_{\min}$ - 1 columns of **H** are linearly independent.

$$d_{\min} \leq n - k + 1 \text{ (Singleton Bound)}$$

Maximal Separable Code:

$d_{\min} = n - k + 1$, e.g. Reed-Solomon Code

- Some Specific Linear Block Codes
  - Hamming Code (n, k, dmin) = (2m - 1, 2m - 1 - m, 3)
  - Hadamard Code (n, k, dmin) = (2m, m + 1, 2m - 1)

Easy Encoding

- Cyclic Codes
  If $\mathbf{C} = (c_{n-1}, \ldots, c_0)$ is a codeword
  $\rightarrow (c_{n-2}, \ldots, c_0, c_{n-1})$ is also a codeword.
- Codeword polynomial: $C(p) = c_{n-1} p^{n-1} + \ldots + c_n$

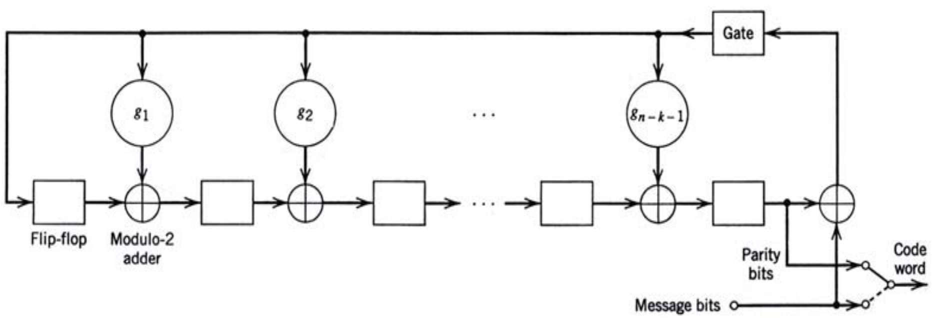
$$pC(p) \bmod p^n - 1 \leftarrow \text{CyclicShift}$$

Encoding: Message polynomial
$X(p) = x_{k-1}\, p^{k-1} + \cdots + x_n \rightarrow$
Codeword polynomial $C(p) = X(p)g(p)$
where $g(p)$: generator polynomial of degree $n-k$

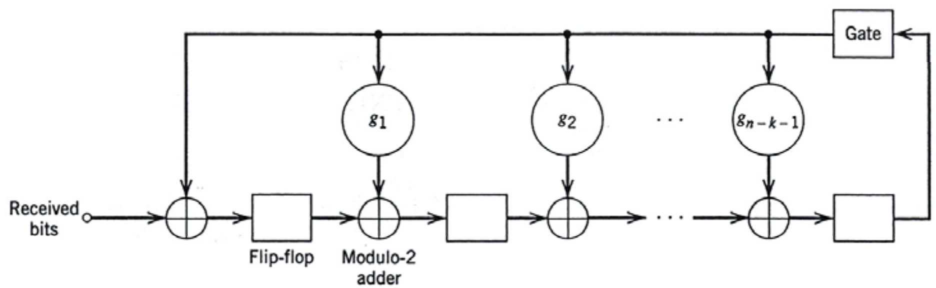
$$p^n + 1 = g(p)h(p)$$

$h(p)$: Parity polynomial

Encoder is implemented by Shift registers.



Encoder for an (n, k) cyclic code.



Syndrome calculator for (n, k) cyclic code.

Digital to Analog (BPSK)

$$c=1 \quad \to \quad s=+1$$
$$0 \quad \to \quad s=-1$$
$$\therefore \quad s=2c-1$$

Soft-Decoding & Maximum Likelihood

$$\begin{aligned}\mathbf{r} &= \mathbf{s}^{(k)}+\mathbf{n}\\ &= (r_1,\cdots,r_n)\\ &= (s_1,\cdots,s_n)+(n_1,\cdots,n_n)\end{aligned}$$

$$\text{Prob}\left[\mathbf{r}\middle|\mathbf{s}^{(k)}\right] : \text{Likelihood}$$

$$\underset{k}{\text{Max}}\ \text{Prob}\left[\mathbf{r}\middle|\mathbf{s}^{(k)}\right] \to \underset{k}{\text{Min}}\left(\mathbf{r}-\mathbf{s}^{(k)}\right)^2$$

$$\to \underset{k}{\text{Max}} : \text{Correlation}\left[\mathbf{r},\mathbf{c}^{(k)}\right]$$

- Optimum Soft-Decision Decoding of Linear Block Codes
  Optimum receiver has $M=2^k$ Matched Filter $\to$ M correlation metrics

$$C(\mathbf{r},\mathbf{C}_i)=\sum_{j=1}^{n}(2c_{ij}-1)r_j$$

where $\mathbf{C}_i$: i-th codeword

c_{ij} : j-th position bit of the i-th codeword

r_j : j-th received signal

$\to$ Largest matched filter output is selected.

Error probability for soft-decision decoding (Coherent PSK)

$$P_M < \exp(-\gamma_b R_c d_{\min} + k\ln 2)$$

where γ_b : SNR per bit

R_c : Coding rate $(=k/n)$

Uncoded binary PSK

$$P_e < \frac{1}{2}\exp(-\gamma_b)$$

Coding gain:

$$Cg = 10\log\left(R_c d_{\min} - k\ln 2/\gamma_b\right)$$

$$d_{\min}\uparrow \quad \rightarrow \quad Cg\uparrow$$

- Hard-Decision Decoding

  Discrete-time channel =

  modulator + AWGN channel + demodulator

  → BSC with crossover probability (p)

$$p = Q\left(\sqrt{2\gamma_b R_c}\right): \text{coherent PSK}$$

$$Q\left(\sqrt{\gamma_b R_c}\right): \text{coherent FSK}$$

$$\tfrac{1}{2}\exp\left(-\tfrac{1}{2}\gamma_b R_c\right): \text{noncoherent FSK}$$

Maximum-Likelihood Decoding →
Minimum Distance Decoding
Syndrome Calculation by Parity check matrix **H**

$$\begin{aligned}\mathbf{S} &= \mathbf{Y}\mathbf{H}^t\\ &= \left(\mathbf{C}_m + \mathbf{e}\right)\mathbf{H}^t\\ &= \mathbf{e}\mathbf{H}^t\end{aligned}$$

where $\mathbf{C}_m$: transmitted codeword

$\mathbf{Y}$: received codeword at the demodulator

$\mathbf{e}$: binary error vector

- Comparison of Performance between **Hard-Decision** and **Soft-Decision** Decoding

→ At most $\approx$2dB difference

- **Bounds on Minimum Distance of Linear Block Codes (R_c vs. d_{min})**
  - Hamming upper bound ($2t < d_{\min}$)

$$1 - R_c \geq \frac{1}{n}\log_2 \sum_{i=0}^{t} \binom{n}{i}$$

  - Plotkin upper bound

$$\frac{d_{\min}}{n}\left(1 - \frac{1}{2d_{\min}}\log_2 d_{\min}\right) \leq \frac{1}{2}\left(1 - R_c + \frac{2}{n}\right)$$

  - Elias upper bound

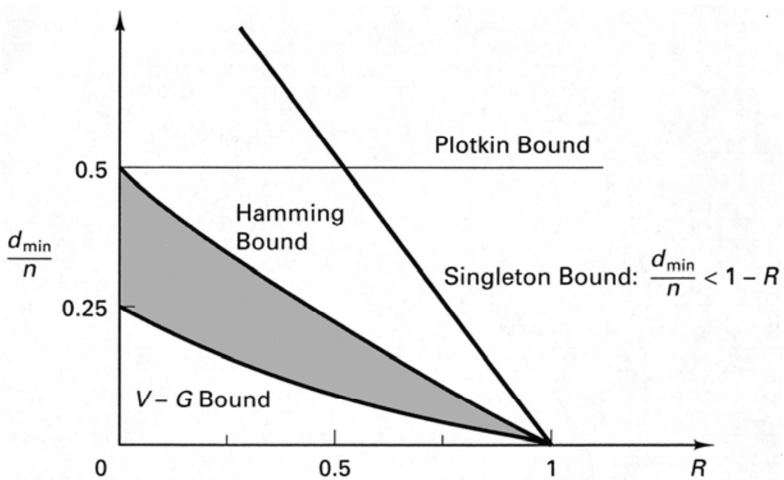
$$\frac{d_{\min}}{n} \leq 2A(1-A)$$

$$R_c = 1 + A\log_2 A + (1-A)\log_2(1-A)$$

  - Gilbert-Varsharmov lower bound

$$\frac{d_{\min}}{n} \geq \alpha$$

$$R_c = 1 - H(\alpha)$$



- Interleaving of Coded Data for Channels with Burst Errors
  Multipath and fading channel → burst error
  Burst error correction code: **Fire code**
  Correctable burst length b

$$b \leq \left\lfloor \frac{1}{2}(n-k) \right\rfloor$$

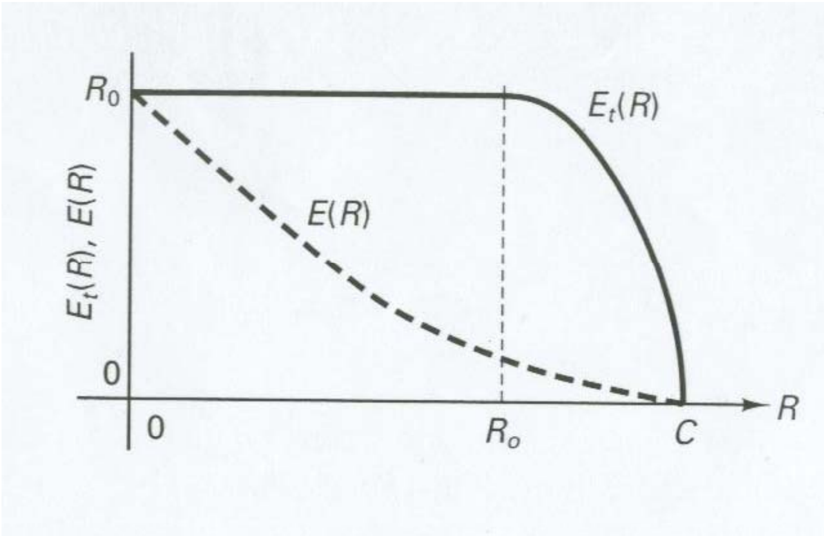
  Block and Convolution interleave is effective for burst error.

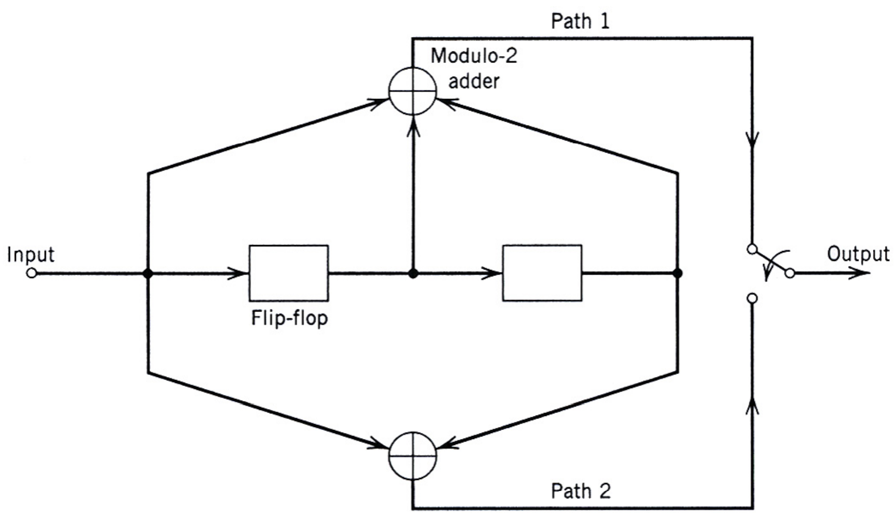
Convolution Codes

Performance of **convolution code > block code** shown by Viterbi's Algorithm.

$$\overline{P(e)} \le e^{-nE(R)}$$

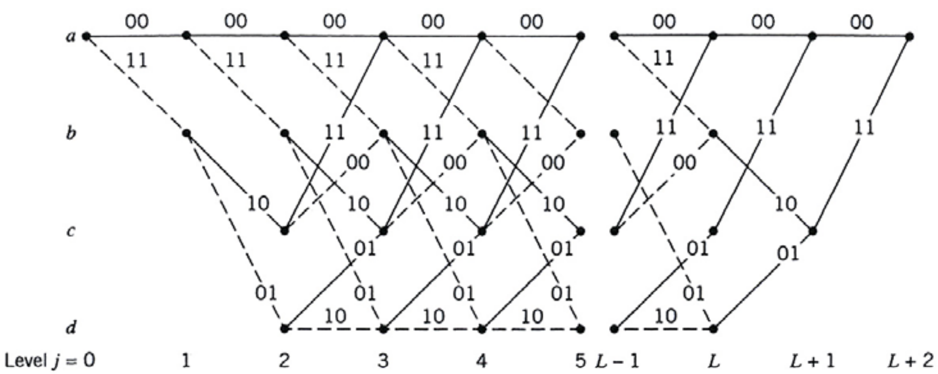
E(R) : Error Exponent





Constraint length-3, rate-1/2 convolutional encoder.

- Parameter of convolution code:
  Constraint length, K
  Minimum free distance
- Optimum Decoding of Convolution Codes – The Viterbi Algorithm
  For $K \le 10$, this is practical.
- Probability of Error for Soft-Decision Decoding



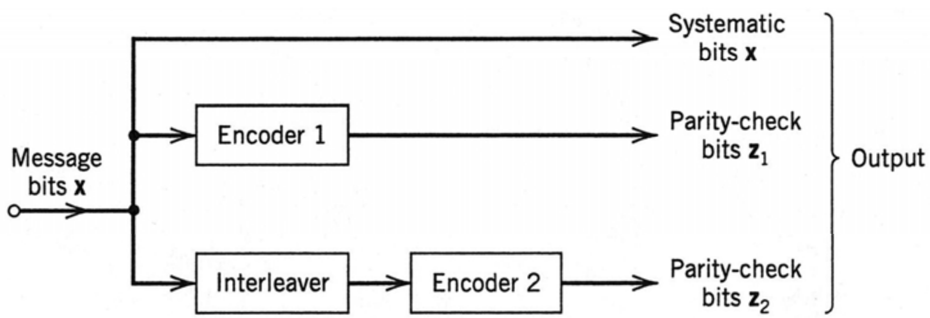
Trellis for the convolutional encoder

$$P_e \le \sum_{d=d_{\text{free}}}^{\infty} a_d Q\left(\sqrt{2\gamma_b R_c d}\right)$$

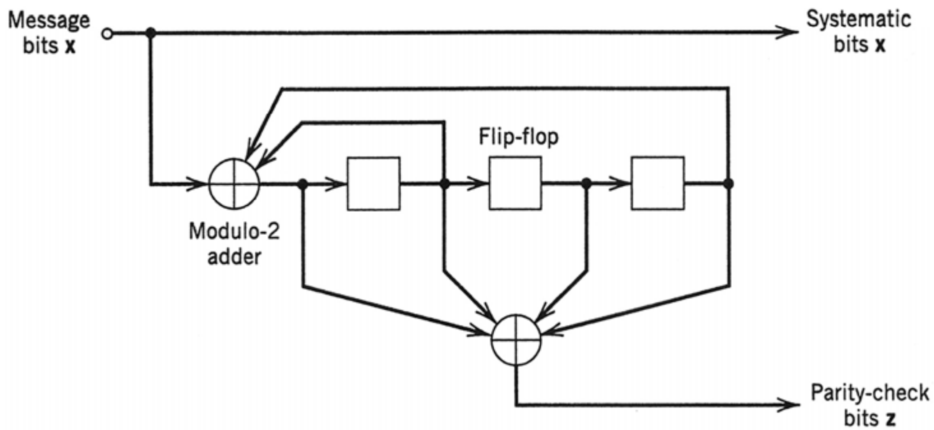
where a_d: the number of paths of distance d

- Probability of Error for Hard-Decision Decoding Hamming distance is a metric for hard-decision

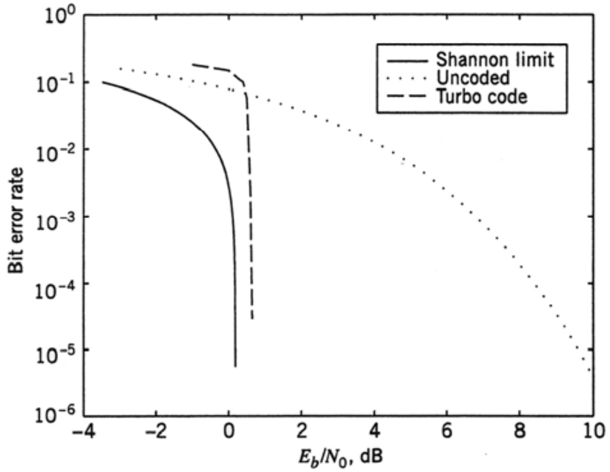
Turbo Coding



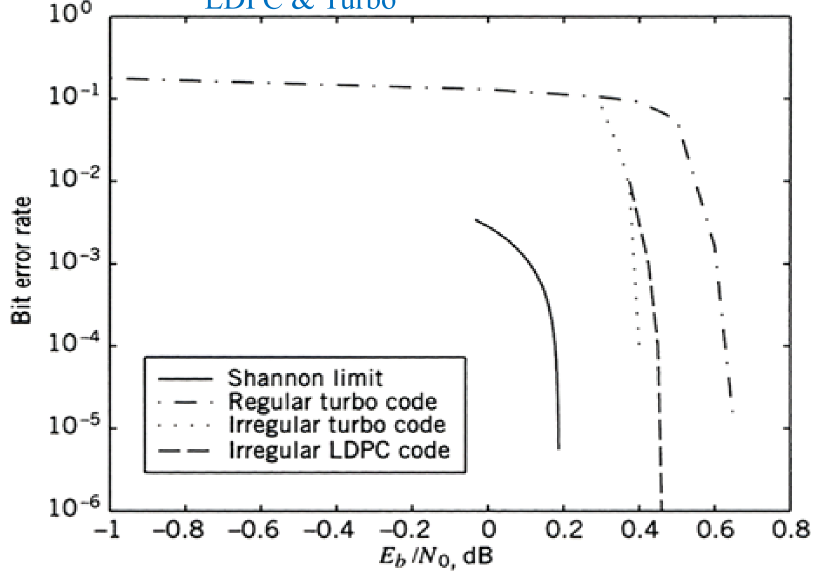
RSC Encoder



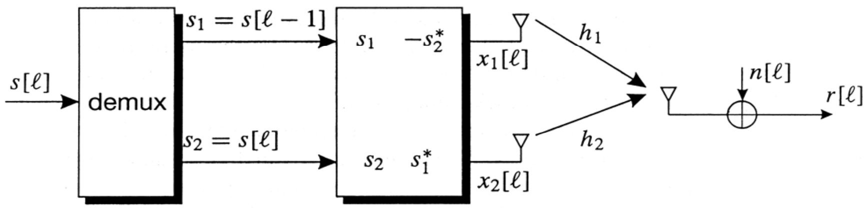
Shannon Limit & Turbo Code



LDPC & Turbo



Alamouti's space–time block code



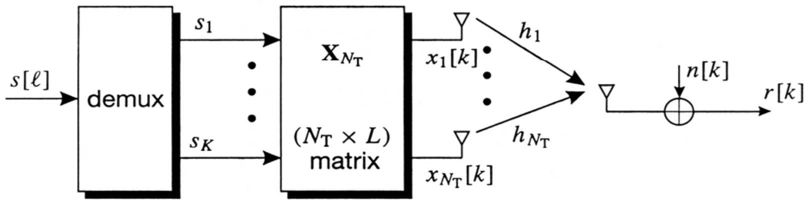
- Received block of two consecutive symbols

$$\tilde{\mathbf{r}} = \begin{pmatrix} r[\ell] \\ r[\ell+1]^* \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} n[\ell] \\ n[\ell+1]^* \end{pmatrix} = \tilde{\mathbf{H}} \cdot \mathbf{s} + \tilde{\mathbf{n}}$$

- Estimated symbol vector after matched filtering

$$\mathbf{y} = \tilde{\mathbf{H}}^H \cdot \tilde{\mathbf{r}} = \frac{1}{2} \cdot \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{pmatrix} \cdot \mathbf{s} + \tilde{\mathbf{H}}^{\mathrm{H}} \cdot \tilde{\mathbf{n}}$$

Orthogonal space–time block codes



- Code rate

$$R = \frac{K}{L}$$

- Spectral efficiency

$$\eta = m \cdot R = m \cdot \frac{K}{L}$$

- Orthogonality constraint

$$\mathbf{X}_{N_\mathrm{T}} \mathbf{X}_{N_\mathrm{T}}^H = \frac{K}{N_\mathrm{T}} \cdot \frac{E_\mathrm{s}}{T_\mathrm{s}} \cdot \mathbf{I}_{N_\mathrm{T}}$$

Orthogonal space–time block codes for $R = 1/2$

- $N_\mathrm{T} = 3$ transmit antennas ($L = 8,\ K = 4$)

$$\mathbf{X}_3 = \frac{1}{\sqrt{6}} \cdot \begin{pmatrix} s_1 & -s_2 & -s_3 & -s_4 & s_1^* & -s_2^* & -s_3^* & -s_4^* \\ s_2 & s_1 & s_4 & -s_3 & s_2^* & s_1^* & s_4^* & -s_3^* \\ s_3 & -s_4 & s_1 & s_2 & s_3^* & -s_4^* & s_1^* & s_2^* \end{pmatrix}$$

- $N_\mathrm{T} = 4$ transmit antennas ($L = 8,\ K = 4$)

$$\mathbf{X}_4 = \frac{1}{\sqrt{8}} \cdot \begin{pmatrix} s_1 & -s_2 & -s_3 & -s_4 & s_1^* & -s_2^* & -s_3^* & -s_4^* \\ s_2 & s_1 & s_4 & -s_3 & s_2^* & s_1^* & s_4^* & -s_3^* \\ s_3 & -s_4 & s_1 & s_2 & s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & s_3 & -s_2 & s_1 & s_4^* & s_3^* & -s_2^* & s_1^* \end{pmatrix}$$

Orthogonal space–time block codes for $R = 3/4$

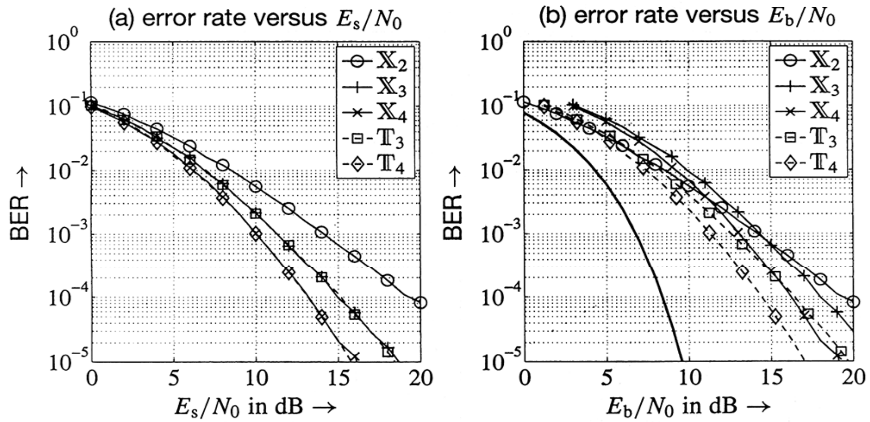
- $N_\mathrm{T} = 3$ transmit antennas ($L = 4,\ K = 3$)

$$\mathbf{T}_3 = \frac{1}{\sqrt{12}} \cdot \begin{pmatrix} 2s_1 & -2s_2^* & \sqrt{2}s_3^* & \sqrt{2}s_3^* \\ 2s_2 & 2s_1^* & \sqrt{2}s_3^* & -\sqrt{2}s_3^* \\ \sqrt{2}s_3 & \sqrt{2}s_3 & -s_1 - s_1^* + s_2 - s_2^* & s_1 - s_1^* + s_2 + s_2^* \end{pmatrix}$$

- $N_\mathrm{T} = 4$ transmit antennas ($L = 4,\ K = 3$)

$$\mathbf{T}_4 = \frac{1}{4} \begin{pmatrix} 2s_1 & -2s_2^* & \sqrt{2}s_3^* & \sqrt{2}s_3^* \\ 2s_2 & 2s_1^* & \sqrt{2}s_3^* & -\sqrt{2}s_3^* \\ \sqrt{2}s_3 & \sqrt{2}s_3 & -s_1 - s_1^* + s_2 - s_2^* & s_1 - s_1^* + s_2 + s_2^* \\ \sqrt{2}s_3 & -\sqrt{2}s_3 & -s_1 - s_1^* - s_2 - s_2^* & -(s_1 + s_1^* + s_2 + s_2^*) \end{pmatrix}$$

Performance of orthogonal space–time block codes for BPSK



Network Coding

- A →[X]→ B
- B →[X]→ C
- B ←[Y]← C
- A ←[Y]← B

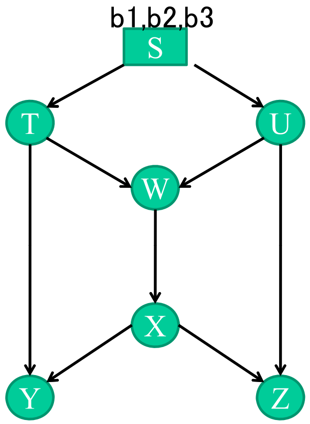
---- 4 steps ----

- A →[X]→ B
- B ←[Y]← C
- A←[X＋Y]←B→[X＋Y]→C

---- 3 steps ----

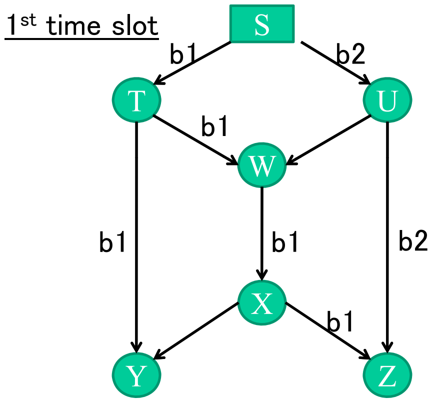
A famous example – butterfly network

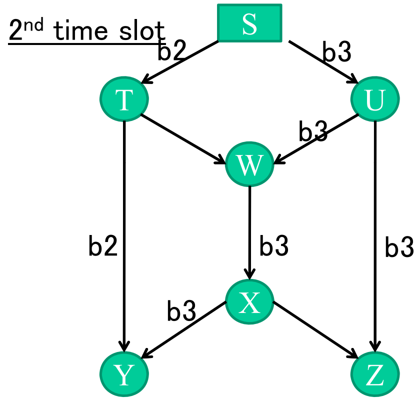
- S: source node
- T, U, W, X: relay nodes
- Y, Z: destination
- S needs to send 3 bits b1, b2, b3 to both Y and Z (multicast)
- Link capacity is 1



A famous example – butterfly network without network coding

- Simple store and forward
- Multicast rate of 1.5 bits per time unit



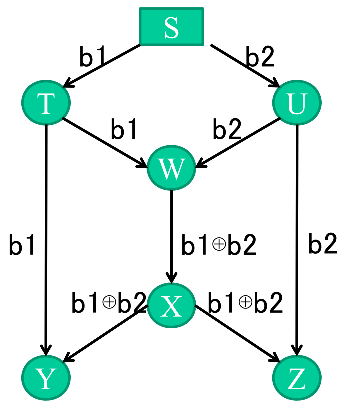


A famous example – butterfly network with network coding

- W receives b1 and b2, then performs exclusive OR (XOR) on received bits and forward to X
- Y receives b1 and b1⊕b2, then extracts b2 as

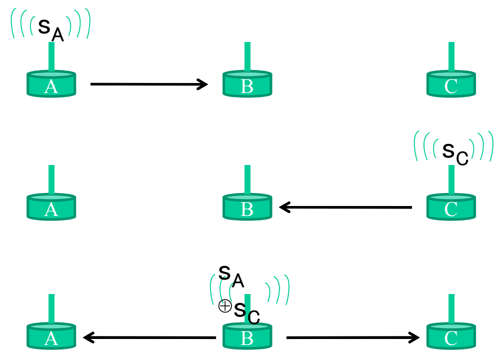
  $b2 = b1 \oplus (b1 \oplus b2)$
- Z receives b2 and b1⊕b2 then extracts b1 as

  $b1 = b2 \oplus (b1 \oplus b2)$
- Achieve multicast rate of 2 bits per time unit



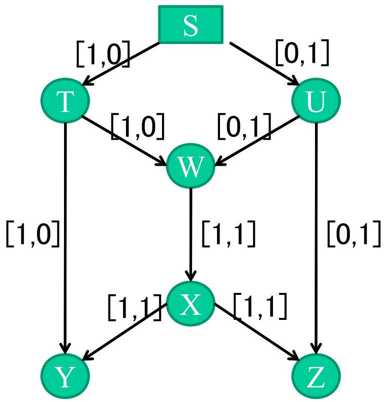
Network coding in wireless environment

- It is easy to apply network coding in wireless environment owing to the broadcast characteristics
- An example shows data from A to C and from C to A can be relayed through B efficiently using network coding



Network coding header

- In network coding, since information is processed inside the network, network coding header is required for network decoding at the destination
- Network coding header describes how a packet is processed
- The right figure shows network coding header of the butterfly network example
- If packet length is long enough, we can neglect the inefficiency of header



Multi-user Communications

Multiple Access Techniques

1. A common communication channel is shared by many users.
   up-link in a satellite communication, a set of terminals →
   a central computer, a mobile cellular system
2. A broadcast network
   down-links in a satellite system, radio and TV broadcast systems
3. Store-and-forward networks
4. Two-way communication systems

-FDMA (Frequency-division Multiple Access)
-TDMA (Time-division Multiple Access)
-CDMA (Code-division Multiple Access):
for burst and low-duty-cycle information transmission

Spread spectrum signals → small cross-correlations
For no spread random access, collision and interference occur.
Retransmission Protocol

Capacity of Multiple Access Methods

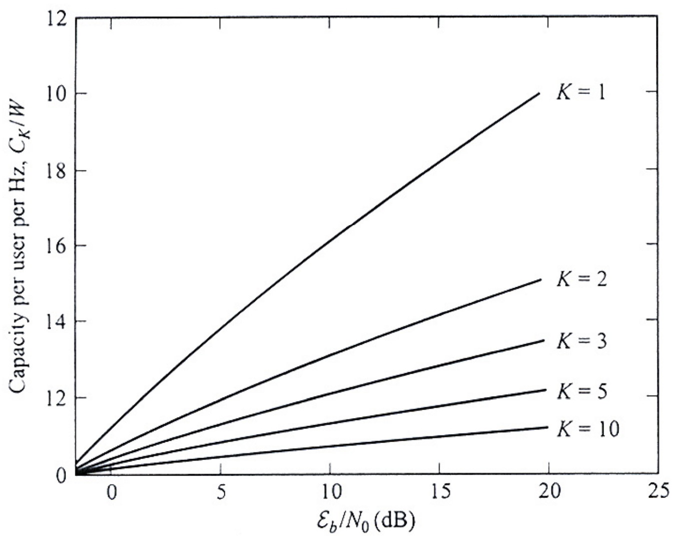
In FDMA, normalized total capacity $C_n = KC_K / W$
(total bit rate for all K users per unit of bandwidth)

$$C_n = \log_2\left(1 + C_n \frac{E_b}{N_0}\right)$$

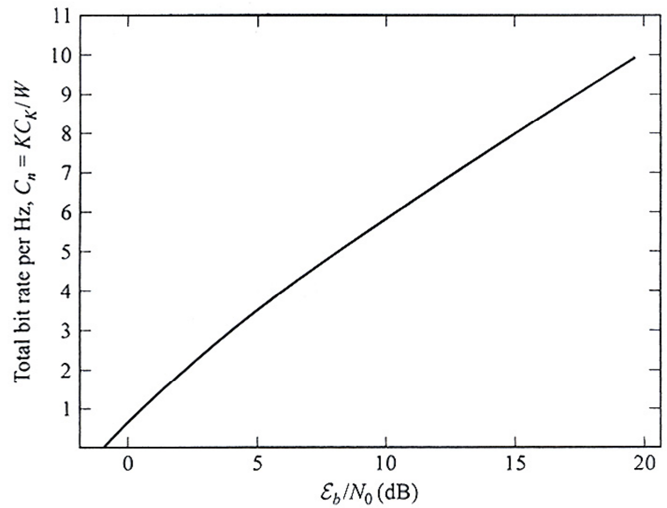
where W : Bandwidth

E_b : Energy per bit

N_0 : Noise power spectrum desity



Normalized capacity as a function of ε_b / N_0 for FDMA.

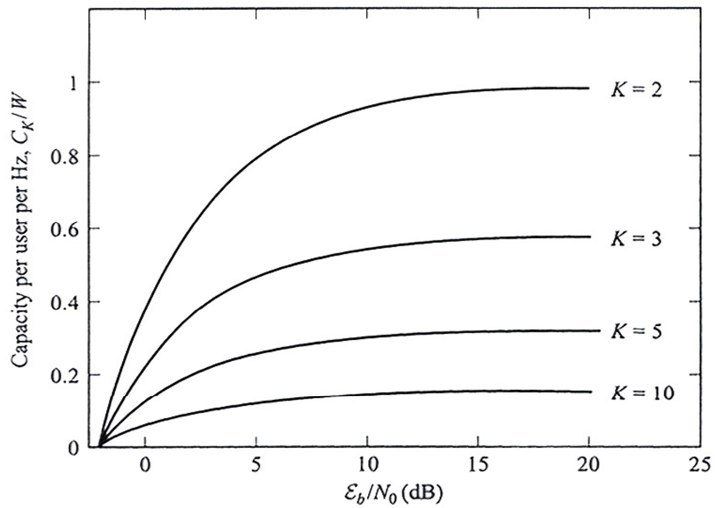


Total capacity per hertz as a function of ε_b / N_0 for FDMA.

In TDMA, there is a practical limit for the transmitter power

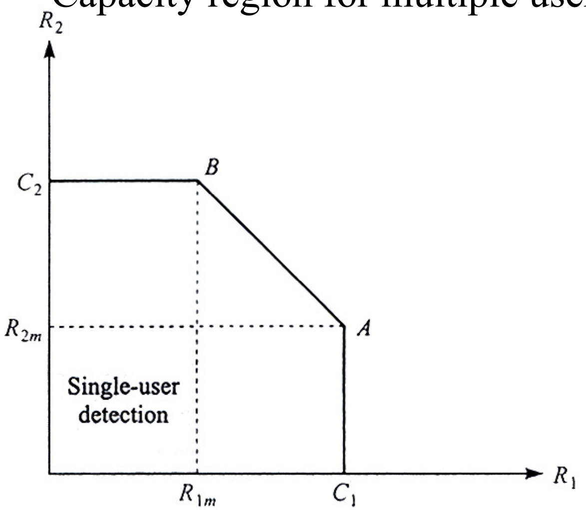
In no cooperative CDMA,

$$C_n \le \log_2 e - \frac{1}{E_b / N_0}$$



Normalized capacity as a function of ε_b / N_0 for noncooperative CDMA.

Capacity region for multiple users



Capacity region of two-user CDMA multiple access Gaussian channel.

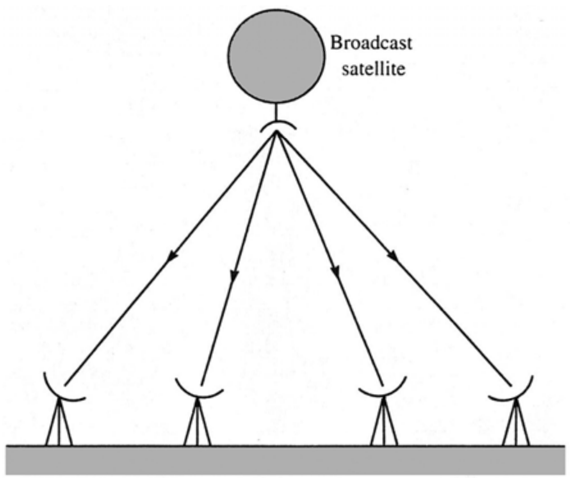
Code-Division Multiple Access

- CDMA Signal and Channel Models
- The Optimum Receiver
  Synchronous Transmission
  Asynchronous Transmission

- Sub-optimum Detectors
  Computational complexity grows linearly with the number of users, *K*.
  Conventional Single-user Detector
  Near-far problem
  Decorrelation Detector
  Minimum Mean-Square-Error Detector
  Other Types of Detectors
- Performance Characteristics of Detectors

Random Access Methods

- ALOHA Systems and Protocols Channel access protocol
  Synchronized (slotted) ALOHA
  Unsynchronized (un-slotted) ALOHA
  Throughput for slotted ALOHA



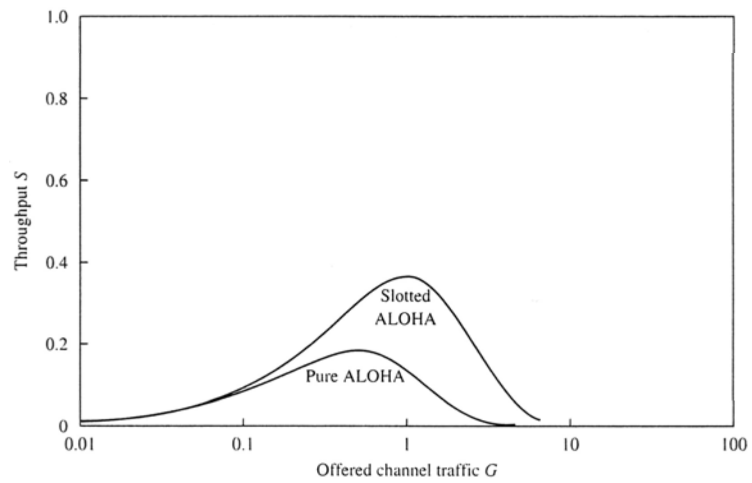
Packet Transmission

- Poisson Point Process: The start time of packets
- Average rate : λ [packets/s]
- Time duration of a packet : Tp
- Offered channel traffic : $G=\lambda Tp$

Throughput Performance

- Unsynchronized random access:
  S=Gexp(-2G)
  Smax=1/(2e)=0.184(packets/slot) @G=1/2
- Slotted ALOHA:
  S=Gexp(-G) for K→∞
  Smax=1/e=0.368(packets/slot) @G=1

Throughput & Delay Performance



- Carrier Sense Systems and Protocols
  CSMA / CD
  (carrier sense multiple access with collision detection)
  No persistent CSMA
  1-persistent CSMA
  p-persistent CSMA

Nonpersistent CSMA

- (a) If the channel is idle, the user transmits a packet.
- (b) If the channel is sensed busy, the user schedules the packet transmission at a later time according to some delay distribution. At the end of the delay interval, repeats steps(a) and (b).

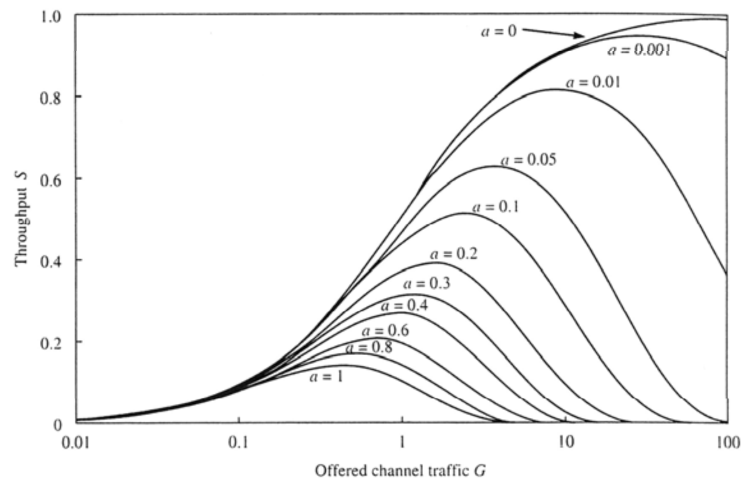
1-persistent CSMA

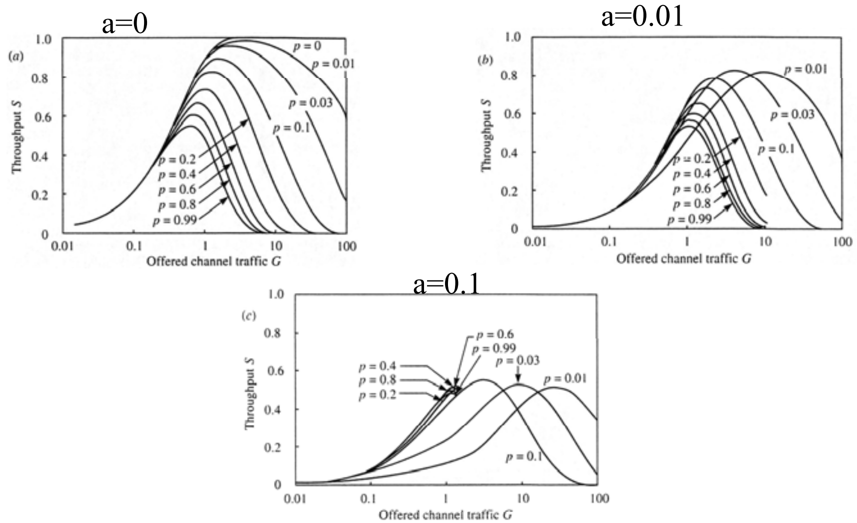
- (a) If the channel is sensed idle, the user transmits the packet with probability 1.
- (b) If the channel is sensed busy, the user waits until the channel becomes idle and transmits a packet with probability one. Note that in this protocol, a collision will always occur when one user has a packet to transmit.

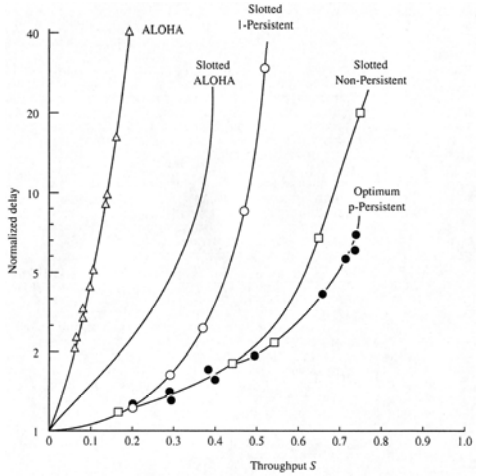
p-persistent CSMA

- (a) If the channel is idle, the packet is transmitted with probability p, and with probability 1-p the transmission is delayed by τ.
- (b) If at t=τ, the channel is still sensed to be idle, step (a) is repeated. If a collision occurs, the user schedules retransmission of the packets according to some preselected transmission delay distribution.
- (c) If at t=τ, the channel is sensed busy, the user waits until it becomes idle, and then operates as in (a) and (b) above.

a=Tp/τd Nonpersistent CSMA






ALOHA
Slotted
1-Persistent
Slotted
ALOHA
Slotted
Non-Persistent
Optimum
p-Persistent
Normalized delay
Throughput S
40
20
10
5
2
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0