Time Integration

After discretization in space, partial differential equation reduces to a ordinal differential equation in time.

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

Ordinal diff. eq.

$$\frac{\partial f}{\partial t} = -u_j \frac{f_j - f_{j-1}}{\Delta x}$$

Taylor-expansion series

$$f(t^{n+1}) = f(t^n + \Delta t)$$
$$= f(t^n) + \left.\frac{df}{dt}\right|_{t=t^n} \Delta t + \frac{1}{2}\left.\frac{d^2 f}{dt^2}\right|_{t=t^n} \Delta t^2 + \frac{1}{6}\left.\frac{d^3 f}{dt^3}\right|_{t=t^n} \Delta t^3 \cdots$$

1st order approximation: $\frac{f^{n+1} - f^n}{\Delta t} = \left.\frac{\partial f}{\partial t}\right|^n$

Advection equation

$$\frac{\partial f}{\partial t} = -u\frac{\partial f}{\partial x} \quad \Rightarrow \quad f^{n+1} = f^n - u\left.\frac{\partial f}{\partial x}\right|^n \Delta t$$

Semi-Lagrangian Scheme(1/2)

Advection equation : relation between time and space

$$\frac{\partial f}{\partial t} = -u\frac{\partial f}{\partial x} \qquad \frac{\partial^2 f}{\partial t^2} = u^2\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^3 f}{\partial t^3} = -u^3\frac{\partial^3 f}{\partial x^3}$$

Substituting into the Taylor-expansion series,

$$f^{n+1} = f^n - u\left.\frac{\partial f}{\partial x}\right|^n \Delta t + \frac{1}{2}u^2\left.\frac{\partial^2 f}{\partial x^2}\right|^n \Delta t^2 + \frac{1}{6}u^3\left.\frac{\partial^3 f}{\partial x^3}\right|^n \Delta t^3 + \cdots$$
$$= f^n - \left.\frac{\partial f}{\partial x}\right|^n (u\Delta t) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|^n (u\Delta t)^2 + \frac{1}{6}\left.\frac{\partial^3 f}{\partial x^3}\right|^n (u\Delta t)^3 + \cdots$$

Semi-Lagrangian Scheme(2/2)

Interpolation function in local region:

$$F(x) = ax^3 + bx^2 + cx + f_j$$
$$\left.\frac{\partial^3 f}{\partial x^3}\right|_j = 6a \qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_j = 2b \qquad \left.\frac{\partial f}{\partial x}\right|_j = c$$

Substituting into the Taylor-expansion series,

$$f^{n+1} = f^n - c(u\Delta t) + b(u\Delta t)^2 - a(u\Delta t)^3 + \cdots$$
$$= F(-u\Delta t)$$

CIP / Cubic Semi-Lagrangian operation

= 3rd-order time integration

Predictor-Corrector

$f(t)$ is a dependent variable of t ,

Ordinal differential equation: $\dfrac{df}{dt} = S(y,t)$

Integrating from t^n to t^{n+1} ,

$$f^{n+1} = f^n + \int_{t^n}^{t^{n+1}} S(f,t)dt$$

The integration can not be done explicitly, because f is included in the non-analytical function S.

Definition : $f^n \equiv f(t^n)$, $S^n \equiv S(f^n, t^n)$

Prediction Phase

By using the n-step (current step) f^n value and r-numbers previous time-step values, $f^{n-1}, f^{n-2}, \cdots f^{n-r}$,
Approximate function of $S(f,t)$ can be constructed.

Using f^n and f^{n-1} , the 1st-order fucntion :

$$S_p^1 = \frac{t - t^{n-1}}{\Delta t} S^n - \frac{t - t^n}{\Delta t} S^{n-1} + O(\Delta t)$$

By substituting into the integration, we have

$$f_p^{n+1} = f^n + \int_{t^n}^{t^{n+1}} S_p^1(f,t)dt$$
$$= f^n + \frac{\Delta t}{2}\left(3S^n - S^{n-1}\right) + O(\Delta t^2)$$

Higher-order Prediction

Using the more previous values, the more accurate predictions are obtained

$r = 2$: $y_p^{n+1} = y^n + \frac{\Delta t}{12}\left(23 f^n - 16 f^{n-1} + 5 f^{n-2}\right) + O(\Delta t^3)$

$r = 3$: $y_p^{n+1} = y^n + \frac{\Delta t}{24}\left(55 f^n - 59 f^{n-1} + 37 f^{n-2} - 9 f^{n-3}\right) + O(\Delta t^4)$

$r = 4$: $y_p^{n+1} = y^n + \frac{\Delta t}{720}\left(1901 f^n - 2774 f^{n-1} + 2616 f^{n-2} - 1274 f^{n-3} + 251 f^{n-4}\right) + O(\Delta t^5)$

These are called Adams – Bashforth formula.
(J.C.Adams – F.Bashforth)

Correction Phase

Getting f_p^{n+1} enables us to construct $S^{n+1} = S\left(f_p^{n+1}, t^{n+1}\right)$.
Using $f^{n+1}, f^n, f^{n-1}, \cdots f^{n-r+1}$,
r-th-order polynomial function can be constructed.

1st-order approximate function:

$$S_c^1 = \frac{t - t^n}{\Delta t} S^{n+1} + \frac{t^{n+1} - t}{\Delta t} S^n + O(\Delta t)$$

By substituting into the integration, we have the corrected value.

$$f_c^{n+1} = f^n + \int_{t^n}^{t^{n+1}} S_c^1(y,t)dt$$
$$= f^n + \frac{\Delta t}{2}\left(S^{n+1} + S^n\right)$$

Higher-order Correction

$r=2: \quad y_c^{n+1} = y^n + \frac{\Delta t}{12}\left(5 f^{n+1} + 8 f^n - f^{n-1}\right)$

$r=3: \quad y_c^{n+1} = y^n + \frac{\Delta t}{24}\left(9 f^{n+1} + 19 f^n - 5 f^{n-1} + f^{n-2}\right)$

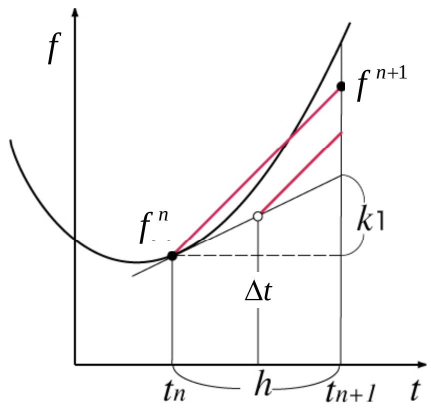
$r=4: \quad y_c^{n+1} = y^n + \frac{\Delta t}{720}\left(251 f^{n+1} + 646 f^n - 264 f^{n-1} + 106 f^{n-2} - 19 f^{n-3}\right)$

These are called the Adams–Moulton formula.
(J.C.Adams – F.R.Moulton)

Repetition of the prediction-correction loop improves the accuracy of the convergence.

Runge-Kutta Method

$$\frac{df}{dt} = S(y,t)$$



1-stage RK Method (1st-Order Euler)

$$f^{n+1} = f^n + \Delta t\, S(f^n, t^n) + \underline{O(\Delta t^2)}$$

2-stage RK Method

$$k_1 = \Delta t\ S(x_n, t_n)$$

$$f^{n+1} = f^n + \Delta t\, S(f^n + 1/2 k_1, t^n + 1/2 \Delta t) + \underline{O(\Delta t^3)}$$

$O(\Delta t^2)$ **error can be eliminated.**

4-Stage Runge-Kutta

By eliminating higher-order error, The following 4-Stage Runge-kutta method is derived.

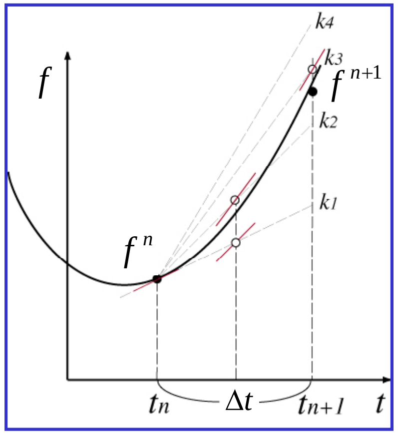
$$k_1 = \Delta t\, S(f^n, t^n)$$

$$k_2 = \Delta t\, S(f^n + 1/2 k_1, t^n + 1/2 \Delta t)$$

$$k_3 = \Delta t\, S(f^n + 1/2 k_2, t^n + 1/2 \Delta t)$$

$$k_4 = \Delta t\, S(f^n + k_3, t^n + \Delta t)$$

$$f^{n+1} = f^n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} + \underline{O(\Delta t^5)}$$



Up to 4 stage, the stage number can reach the accuracy order.

Reference

3-stage RK Method (3rd-order accuracy)

$$k_1 = \Delta t\, S(f^n, t^n)$$

$$k_2 = \Delta t\, S(f^n + \frac{2}{3} k_1, t^n + \frac{2}{3} \Delta t)$$

$$k_3 = \Delta t\, S(f^n + \frac{2}{3} k_2, t^n + \frac{2}{3} \Delta t)$$

$$f^{n+1} = f^n + \frac{2k_1 + 3k_2 + 3k_3}{8} + O(\Delta t^4)$$

2D Advection Equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0$$

Discretization : 1st-order Upwind scheme

For $u \ge 0, \quad v \ge 0$ $\quad f_{i,j}^{n+1} = f_j^n - u\frac{f_{i,j}^n - f_{i-1,j}^n}{\Delta x}\Delta t - v\frac{f_{i,j}^n - f_{i,j-1}^n}{\Delta y}$

TEST PRLOBLEM:

Computational Domain

$0 \le x \le 1, \quad 0 \le y \le 1$

Periodic Boundary Condition

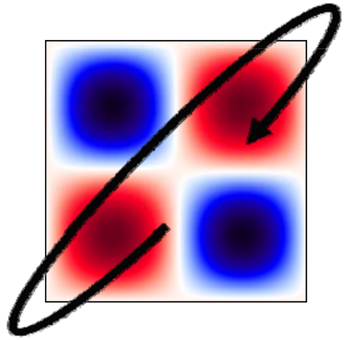
Initial Condition : $f(x,y) = \sin(k_x x)\sin(k_y y)$

$\left(k_x = 2\pi, \quad k_y = 2\pi\right)$

Accuracy Check

At $t = 1.0$ **for** $u = 1.0, \quad v = 1.0$

The profile comes back to the initial one.



Source Code