

3. Direct Integration of the Field Equations

◆ *Vector equivalent theorem* ∇ は積分座標系

$$\text{公式} \quad \int_V (Q \cdot \nabla \times \nabla \times P - P \cdot \nabla \times \nabla \times Q) dv = \oint_S (P \times \nabla \times Q - Q \times \nabla \times P) \cdot n da \quad (A1)$$

Vector Algebra ベクトル解析の公式 n : 外向き

V, U scalar スカラ量 A, B vector ベクトル量

A) ベクトルの微分公式

$$\nabla(VU) = V\nabla U + U\nabla V$$

$$\nabla(A \times B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (B \cdot \nabla)A + (A \cdot \nabla)B$$

$$\nabla \cdot \nabla V = \nabla^2 V$$

$$\nabla \cdot (VA) = V\nabla \cdot A + A\nabla \cdot V$$

$$\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B$$

$$\nabla \cdot \nabla \times A = 0$$

$$\nabla \times (\nabla V) = 0$$

$$\nabla \times (VA) = V\nabla \times A + \nabla V \times A$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\nabla \times (A \times B) = A\nabla \cdot B - B\nabla \cdot A + (B \cdot \nabla)A - (A \cdot \nabla)B$$

◆ *Maxwell's Equations*

Duality of E and H

$$\nabla \times E = -j\omega\mu H - J^*$$

$$E \rightarrow H \quad \varepsilon \leftrightarrow \mu$$

$$\nabla \times H = j\omega\varepsilon E + J$$

$$H \rightarrow -E$$

$$\nabla \cdot H = \frac{\rho^*}{\mu}$$

$$J^* \rightarrow -J$$

$$J \rightarrow J^*$$

$$\nabla \cdot E = \frac{\rho}{\varepsilon}$$

$$\rho \rightarrow \rho^* \quad \rho^* \rightarrow -\rho$$

$$\nabla \cdot J + j\omega\rho = 0$$

$$\nabla \cdot J^* + j\omega\rho^* = 0$$

$$\nabla \times \nabla \times E = -j\omega\mu(j\omega\varepsilon E + J) - \nabla \times J^* = k^2 E - \nabla \times J^* - j\omega\mu J$$

$$\nabla \times \nabla \times H = j\omega\varepsilon(-j\omega\mu H - J^*) + \nabla \times J = k^2 H + \nabla \times J - j\omega\varepsilon J^*$$

Maxwell equation and Vector equivalent theorem $\rightarrow$ Direct Integration

$$\mathbf{P} \equiv \mathbf{E}, \quad \mathbf{Q} \equiv \phi \mathbf{a} \quad \nabla^2 \phi + k^2 \phi = 0 \quad a: \text{定ベクトル}$$

$$\nabla^2 \mathbf{Q} + k^2 \mathbf{Q} = 0$$

Maxwell equation

$$\nabla \times \nabla \times \mathbf{Q} = \mathbf{a} k^2 \phi + \nabla (\mathbf{a} \cdot \nabla \phi) \quad \because \nabla \times (\phi \mathbf{a}) = \phi \nabla \times \mathbf{a} + \nabla \phi \times \mathbf{a} = \nabla \phi \times \mathbf{a}$$

$$\nabla \times \nabla \times \mathbf{P} = k^2 \mathbf{E} - \nabla \times \mathbf{J}^* - j\omega\mu \mathbf{J} \quad \therefore \nabla \times (\nabla \times \mathbf{Q}) = \nabla \phi \nabla \cdot \mathbf{a} - \mathbf{a} \nabla \cdot \nabla \phi + (\mathbf{a} \cdot \nabla) \nabla \phi - (\nabla \phi \cdot) \mathbf{a}$$

Vector equivalent theorem

$$\text{左辺: } \phi \mathbf{a} \cdot (k^2 \mathbf{E} - \nabla \times \mathbf{J}^* - j\omega\mu \mathbf{J}) - \mathbf{E} \cdot (\mathbf{a} k^2 \phi + \nabla (\mathbf{a} \cdot \nabla \phi))$$

$$= \int_v -\phi \mathbf{a} \cdot \nabla \times \mathbf{J}^* - j\omega\mu \phi \mathbf{a} \cdot \mathbf{J} - \mathbf{E} \cdot \nabla (\mathbf{a} \cdot \nabla \phi) dv$$

$$\text{div} \{ \mathbf{E} (\mathbf{a} \cdot \nabla \phi) \} = \mathbf{E} \cdot \nabla (\mathbf{a} \cdot \nabla \phi) + (\mathbf{a} \cdot \nabla \phi) \nabla \cdot \mathbf{E} \quad \text{よ} \cup$$

$$\int \text{div} \mathbf{E} (\mathbf{a} \cdot \nabla \phi) dv = \int \mathbf{E} (\mathbf{a} \cdot \nabla \phi) \cdot d\mathbf{s} = \int \mathbf{E} \cdot \nabla (\mathbf{a} \cdot \nabla \phi) dv + \int (\mathbf{a} \cdot \nabla \phi) \nabla \cdot \mathbf{E} dv$$

$$\therefore \int \mathbf{E} \cdot \nabla (\mathbf{a} \cdot \nabla \phi) dv = \int \mathbf{E} (\mathbf{a} \cdot \nabla \phi) \cdot d\mathbf{s} - \int (\mathbf{a} \cdot \nabla \phi) \frac{\rho}{\epsilon} dv$$

$$\text{左辺} = \int_v \left(-\phi \mathbf{a} \cdot \nabla \times \mathbf{J}^* - j\omega\mu \phi \mathbf{a} \cdot \mathbf{J} + (\mathbf{a} \cdot \nabla \phi) \frac{\rho}{\epsilon} \right) dv - \oint_s \mathbf{E} (\mathbf{a} \cdot \nabla \phi) \cdot d\mathbf{s}$$

$$\text{右辺: } (\mathbf{P} \times \nabla \times \mathbf{Q} - \mathbf{Q} \times \nabla \times \mathbf{P}) \cdot \hat{\mathbf{n}}$$

$$\mathbf{P} \times \nabla \times \mathbf{Q} = \mathbf{E} \times (\nabla \times \phi \mathbf{a}) = \mathbf{E} \times (\phi \nabla \times \mathbf{a} + \nabla \phi \times \mathbf{a}) = \mathbf{E} \times (\nabla \phi \times \mathbf{a}) = \nabla \phi \cdot (\mathbf{E} \cdot \mathbf{a}) - \mathbf{a} \cdot (\nabla \phi \cdot \mathbf{E})$$

$$\mathbf{Q} \times \nabla \times \mathbf{P} = \phi \mathbf{a} \times (-j\omega\mu \mathbf{H} - \mathbf{J}^*)$$

$$\text{右辺} = (\hat{\mathbf{n}} \cdot \nabla \phi) (\mathbf{E} \cdot \mathbf{a}) - (\hat{\mathbf{n}} \cdot \mathbf{a}) (\nabla \phi \cdot \mathbf{E}) + j\omega\mu \phi \mathbf{a} \times \mathbf{H} \cdot \hat{\mathbf{n}} + \phi \mathbf{a} \times \mathbf{J}^* \cdot \hat{\mathbf{n}}$$

$$= \underbrace{\mathbf{a} \cdot \mathbf{E} (\hat{\mathbf{n}} \cdot \nabla \phi) - \mathbf{a} \cdot \hat{\mathbf{n}} (\mathbf{E} \cdot \nabla \phi)}_{\mathbf{a} \cdot (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla \phi} + j\omega\mu \phi \mathbf{a} \cdot (\mathbf{H} \times \hat{\mathbf{n}}) + \phi \mathbf{a} \cdot \mathbf{J}^* \times \hat{\mathbf{n}}$$

$$\text{右辺} = \oint_s \mathbf{a} \cdot (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla \phi + j\omega\mu \phi \mathbf{a} \cdot (\mathbf{H} \times \hat{\mathbf{n}}) + \phi \mathbf{J}^* \times \hat{\mathbf{n}} ds$$

右辺 = 左辺

$$\begin{aligned} \therefore \int \mathbf{a} \cdot \left\{ -\phi \nabla \times \mathbf{J}^* - j\omega\mu\phi \mathbf{J} + \nabla\phi \frac{\rho}{\varepsilon} \right\} dv \\ = \oint \mathbf{a} \cdot \left\{ (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla\phi + j\omega\mu\phi(\mathbf{H} \times \hat{\mathbf{n}}) + \phi \mathbf{J}^* \times \hat{\mathbf{n}} + \nabla\phi(\hat{\mathbf{n}} \cdot \mathbf{E}) \right\} ds \end{aligned}$$

a : 任意 arbitrary

$$\begin{aligned} \int \left(-\phi \nabla \times \mathbf{J}^* - j\omega\mu\phi \mathbf{J} + \nabla\phi \frac{\rho}{\varepsilon} \right) dv \\ = \oint \left\{ j\omega\mu\phi(\mathbf{H} \times \hat{\mathbf{n}}) + (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla\phi + \phi \mathbf{J}^* \times \hat{\mathbf{n}} + \nabla\phi(\hat{\mathbf{n}} \cdot \mathbf{E}) \right\} ds \end{aligned}$$

On the otherhand ところで

$$\oint \phi \mathbf{J}^* \times \hat{\mathbf{n}} ds = - \int_v \left((\phi \nabla \times \mathbf{J}^*) - \mathbf{J}^* \times \nabla\phi \right) dv \quad (\text{A2})$$

$$\begin{aligned} \therefore \int -\mathbf{J}^* \times \nabla\phi - j\omega\mu\phi \mathbf{J} + \nabla\phi \frac{\rho}{\varepsilon} dv \\ = \oint \left\{ j\omega\mu\phi(\mathbf{H} \times \hat{\mathbf{n}}) + (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla\phi + \nabla\phi(\hat{\mathbf{n}} \cdot \mathbf{E}) \right\} ds \end{aligned}$$

Extraction Singularities 特異点の抽出

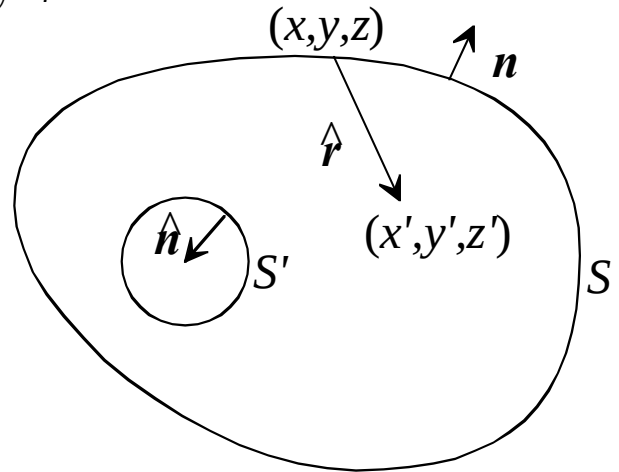
• Definition gradient

(x, y, z) が ρ 方向へ動く r が小さく、 $1/r$ が大きく

$\therefore$ ベクトルは ρ 方向を向いている

$$\text{on } S' \quad \phi = \frac{e^{-jkr}}{r} \quad (x, y, z) \text{系} \rightarrow \nabla \phi = \nabla \left(\frac{e^{-jkr}}{r} \right) = \frac{\hat{\mathbf{n}}}{r^2}$$

$$\begin{aligned} \nabla \phi &= \frac{\partial}{\partial r} \phi(-\hat{\mathbf{n}}) = \frac{-jkr - 1}{r^2} e^{-jkr} (-\hat{\mathbf{n}}) \\ &= \left(\frac{1}{r^2} + jk \frac{1}{r} \right) e^{-jkr} \hat{\mathbf{n}} \\ &\rightarrow \frac{\hat{\mathbf{n}}}{r^2} \end{aligned}$$



$$\int ds \cong 4\pi r^2$$

$$\begin{aligned} \oint_{S'} &\rightarrow \oint_{S'} \left\{ j\omega\mu \frac{1}{r} (\mathbf{H} \times \hat{\mathbf{n}}) + \frac{1}{r^2} [(\hat{\mathbf{n}} \times \mathbf{E}) \times \hat{\mathbf{n}} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{E})] \right\} ds \quad (r \rightarrow 0) \\ &\rightarrow 4\pi \mathbf{E}(x_0, y_0, z_0) \end{aligned}$$

$$\begin{aligned} \therefore \mathbf{E} &= \frac{1}{4\pi} \int_V \left(-j\omega\mu \phi \mathbf{J} - \mathbf{J}^* \times \nabla \phi + \nabla \phi \frac{\rho}{\varepsilon} \right) dv \\ &\quad + \frac{1}{4\pi} \oint_S \left\{ -j\omega\mu \phi (\mathbf{H} \times \hat{\mathbf{n}}) - (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla \phi - \nabla \phi (\hat{\mathbf{n}} \cdot \mathbf{E}) \right\} ds \end{aligned}$$

note :

$$\nabla \phi \frac{\rho}{\varepsilon} = \frac{\nabla \phi \nabla \square}{-j\omega\varepsilon} \quad \because \nabla \square = -j\omega\rho$$

By the way,

$$\mathbf{A} = \frac{\mu}{4\pi} \iiint_V \mathbf{J} \frac{e^{-jkr}}{r} dV$$

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla(\nabla \cdot \mathbf{A})}{j\omega\varepsilon\mu}$$

第1項と第2項と比べて Comparing the first and second terms,

$$\mathbf{J} = \mathbf{H} \times \hat{\mathbf{n}}$$

$$\mathbf{J}^* = \hat{\mathbf{n}} \times \mathbf{E}$$

$$\rho = -\varepsilon(\hat{\mathbf{n}} \cdot \mathbf{E}) \quad \hat{\mathbf{n}}: \text{外向き outer normal}$$

From the duality of $\mathbf{E}$ and $\mathbf{H}$,

$$\begin{array}{lll}
 \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}^* & \mu \rightarrow \varepsilon & \nabla \times (-\mathbf{H}) = -j\omega\varepsilon\mathbf{E} - \mathbf{J} \\
 \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} & \varepsilon \rightarrow \mu & \nabla \times (\mathbf{E}) = j\omega\mu(-\mathbf{H}) - \mathbf{J}^* \\
 \nabla \cdot \mathbf{H} = \frac{\rho^*}{\mu} & \mathbf{E} \rightarrow -\mathbf{H} & \\
 \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} & \mathbf{H} \rightarrow \mathbf{E} & \\
 & \mathbf{J}^* \rightarrow \mathbf{J} & \\
 & \mathbf{J} \rightarrow -\mathbf{J}^* & \\
 & \rho \rightarrow -\rho^* & \\
 & \rho^* \rightarrow \rho &
 \end{array}$$

On the other hand, from the duality

$$\begin{aligned}
 \mathbf{H} &= \frac{1}{4\pi} \int_v \left(-j\omega\varepsilon \mathbf{J}^* \phi + \mathbf{J} \times \nabla \phi + \nabla \phi \frac{\rho^*}{\mu} \right) dv \\
 &+ \frac{1}{4\pi} \oint_s \left\{ -j\omega\varepsilon \phi (\hat{\mathbf{n}} \times \mathbf{E}) - (\hat{\mathbf{n}} \times \mathbf{H}) \times \nabla \phi - \nabla \phi (\hat{\mathbf{n}} \cdot \mathbf{H}) \right\} ds \\
 \mathbf{J}^* &= \hat{\mathbf{n}} \times \mathbf{E} \\
 \mathbf{J} &= \mathbf{H} \times \hat{\mathbf{n}} \\
 \rho^* &= -\mu (\hat{\mathbf{n}} \cdot \mathbf{H})
 \end{aligned}$$

Integral Expression of EM Fields

$$\begin{aligned}
 \mathbf{E} &= \frac{1}{4\pi} \int \left(-j\omega\mu\phi \mathbf{J} - \mathbf{J}^* \times \nabla \phi + \nabla \phi \frac{\rho}{\varepsilon} \right) dv \\
 &+ \frac{1}{4\pi} \oint \left\{ -j\omega\mu\phi (\mathbf{H} \times \hat{\mathbf{n}}) - (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla \phi - \nabla \phi (\hat{\mathbf{n}} \cdot \mathbf{E}) \right\} ds \\
 \mathbf{H} &= \frac{1}{4\pi} \int_v \left(-j\omega\varepsilon \mathbf{J}^* \phi + \mathbf{J} \times \nabla \phi + \nabla \phi \frac{\rho^*}{\mu} \right) dv \\
 &+ \frac{1}{4\pi} \oint_s \left\{ -j\omega\varepsilon \phi (\hat{\mathbf{n}} \times \mathbf{E}) - (\hat{\mathbf{n}} \times \mathbf{H}) \times \nabla \phi - \nabla \phi (\hat{\mathbf{n}} \cdot \mathbf{H}) \right\} ds
 \end{aligned}$$

Compare it with 1-13.