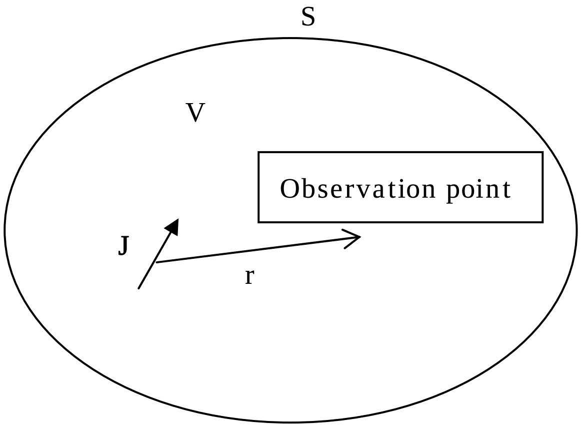
# 1. Radiation from a dipole 波源からの放射界

（**Summary** 概要 まとめ）

**Final results:**

$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J}$

$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$




Vector potential

$\mu\mathbf{H} \equiv \nabla \times \mathbf{A}$

Vector Helmholtz Equation

$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}$

Solution (Reference Carl T. A. Johnk)

$$\mathbf{A} = \frac{\mu}{4\pi}\int \mathbf{J}\frac{e^{-jkr}}{r}dv + \frac{1}{4\pi}\int \frac{\partial}{\partial n}\mathbf{A}\cdot\frac{e^{-jkr}}{r}ds - \frac{1}{4\pi}\int \mathbf{A}\cdot\frac{\partial}{\partial n}\frac{e^{-jkr}}{r}ds$$

If Radiation Condition 放射条件 $r\left\{\frac{\partial}{\partial r}\mathbf{A} + jk\mathbf{A}\right\}_{r=\infty} \to 0$

（No sources/objects outside of S 外に波源、散乱体なし）

Free space Green's Function

$$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkr}}{r}dV$$

Fields are expressed in terms of vector potential **A** only as:

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}$$

$$\mathbf{E} = -j\omega\mathbf{A} + \underline{\frac{\nabla(\nabla\cdot\mathbf{A})}{j\omega\varepsilon\mu}}$$

$\downarrow$

$-\nabla\phi$

When only the **<u>far fields</u>** are considered($|r| \to \infty$), we have a TEM wave.

$$\mathbf{E} = -j\omega\left(\mathbf{A} - \hat{r}\left(\mathbf{A}\cdot\hat{r}\right)\right)$$

$$\mathbf{H} = \frac{1}{\eta}\left(\hat{r}\times\mathbf{E}\right) \quad \eta = \sqrt{\mu/\varepsilon} \approx 120\pi$$

# **<u>Radiation from a dipole</u> (Derivation using potentials).**

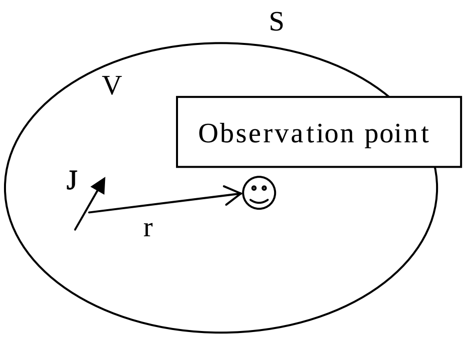
波源からの放射（ポテンシャルによる解法）

Maxwell’s equations:

$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J}$ ①

$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ ②

$\nabla \cdot \mathbf{D} = \rho$ ③




Refer to:

We have to obtain the solution ***E*** for the vector wave equation.波源のあるところでは次のベクトル
方程式の解を求めることになる。

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$$
$$= -j\omega\mu\nabla \times \mathbf{H}$$
$$= k^2\mathbf{E} - j\omega\mu\mathbf{J}$$

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = -j\omega\mu\mathbf{J}$$

From ② and $\nabla \cdot \mathbf{H} = 0$, ②より、$\nabla \cdot \mathbf{H} = 0$であるから

$\mu\mathbf{H} = \nabla \times \mathbf{A}$ ④

where vector potential **A** is introduced.

$\nabla \times \mathbf{E} = -j\omega\nabla \times \mathbf{A}$

$\mathbf{A} = \mathbf{A}_r + \mathbf{A}_p \quad (\nabla \times \mathbf{A}_p = 0)$
$\nabla \times \mathbf{A} = \nabla \times \mathbf{A}_r$
$\mathbf{A}_p$: 任意(*arbitrary*)

We introduce a scalar potential $\phi$:

$(\mathbf{E} + j\omega\mathbf{A}) = -\nabla\phi$

$\mathbf{E} = -j\omega\mathbf{A} - \nabla\phi$ ⑤

④,⑤→ ①

We arrive at the equation for $\mathbf{A}$ and $\phi$.

$$\underline{\nabla \times \nabla \times \mathbf{A}} = j\omega\varepsilon\mu(-j\omega\mathbf{A} - \nabla\phi) + \mu\mathbf{J}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = k^2\mathbf{A} - j\omega\varepsilon\mu\nabla\phi + \mu\mathbf{J}$$

$\nabla^2\mathbf{A} + k^2\mathbf{A} = \nabla(j\omega\varepsilon\mu\phi + \nabla \cdot \mathbf{A}) - \mu\mathbf{J}$ ⑥

The vector potential has the arbitrary divergence component; we can impose **two kinds** of conditions as example.

*Lorentz condition*

$\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\phi = 0$ ⑦

$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$ ⑧

$$\left.\begin{aligned} \mathbf{H} &= \frac{1}{\mu}\nabla \times \mathbf{A} \\ \mathbf{E} &= -j\omega\mathbf{A} + \frac{\nabla(\nabla \cdot \mathbf{A})}{j\omega\varepsilon\mu} \end{aligned}\right\} \quad ⑨$$

⑤, ⑦ → ③

$$\nabla \cdot \mathbf{D} = -j\omega\varepsilon\nabla \cdot \mathbf{A} - \nabla^2\phi\varepsilon$$
$$= -\varepsilon\left(k^2\phi + \nabla^2\phi\right) = \rho$$

$$\nabla^2\phi + k^2\phi = -\frac{\rho}{\varepsilon} \quad ⑩$$

*Helmholtz equation*

*alternative condition*

$\nabla \cdot \mathbf{A} = 0$ ⑦'

$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = j\omega\varepsilon\mu\nabla\phi - \mu \mathbf{J}$ ⑧'

$$\left.\begin{aligned} \mathbf{H} &= \frac{1}{\mu}\nabla \times \mathbf{A} \\ \mathbf{E} &= -j\omega\mathbf{A} - \nabla\phi \end{aligned}\right\} \quad ⑨'$$

⑤, ⑦' → ③

$$\nabla \cdot \mathbf{D} = -j\omega\varepsilon\nabla \cdot \mathbf{A} - \varepsilon\nabla^2\phi$$
$$= -\varepsilon\nabla^2\phi = \rho$$

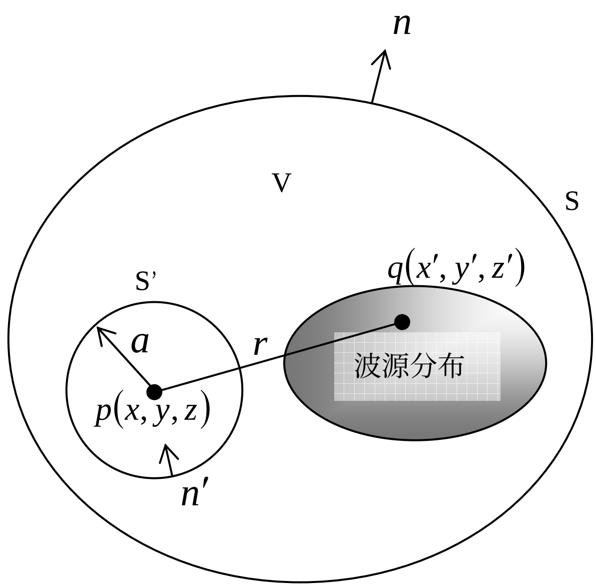
$$\nabla^2\phi = -\frac{\rho}{\varepsilon} \quad ⑩'$$

*Poisson's equation*

## Inhomogeneous Vector Helmholtz equation

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} \qquad ⑧$$

ヘルムホルツの波動方程式




**Solution for the scalar Helmholtz equation**

波動方程式の解

$$\nabla^2 \phi + k^2 \phi = -q \ , \ k^2 = \omega^2 \varepsilon \mu$$

$$\psi = \frac{e^{-jkr}}{r} \qquad r: \textit{from observer}$$

$$q = q(x', y', z')$$

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- *Preliminary information*

$$\begin{cases} \psi : \textit{Solution for} \quad \nabla^2 \psi + k^2 \psi = 0 \quad \rightarrow \textit{check}1[1-6] \\ \textit{Green's theorem} \rightarrow \textit{check}2[1-5] \end{cases}$$

- *Final Results for* ф (*derived later*[1-7])

$$\left.\begin{array}{l} \nabla^2 \phi + k^2 \phi = -q \\ \nabla^2 \psi + k^2 \psi = 0 \end{array}\right\} \rightarrow \iint_v (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dv = \iint_s \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds \quad \textit{(Green's theorem)}$$

$$\iiint_v \left( \phi \left[ -k^2 \psi \right] - \psi \left[ -q - k^2 \phi \right] \right) dv = \iint_{s+s'} \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds$$

(integ.. for $x', y'$ *and* $z'$)

$$\iiint_v \psi q dv = \iint_{S+S'} \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds$$

右辺: $\iint_{S'} \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds \quad$ ($S'$上で)

$$\cong \bar{\phi} \iint_{S'} \frac{e^{-jka}}{a^2} ds - \frac{\partial \bar{\phi}}{\partial n} \iint_{S'} \frac{e^{-jka}}{a} ds \qquad \left( \frac{\partial \psi}{\partial n} = -\frac{\partial \psi}{\partial r} = \frac{jke^{-jkr} r + e^{-jkr}}{r^2} \Rightarrow \frac{e^{-jkr}}{r^2} = \frac{e^{-jka}}{a^2} \right)$$

$$\Rightarrow 4\pi \phi \quad \left( \iint_{S'} ds \Rightarrow 4\pi a^2 \right)$$

| | |
|---|---|
| $\bar{\phi}$: | $S'$上での$\phi$の代表値 |
| $\frac{\partial \bar{\phi}}{\partial n}$: | $S'$上での$\frac{\partial \phi}{\partial n}$の代表値 |

$$\therefore \phi(x, y, z) = \frac{1}{4\pi} \iiint_V \frac{e^{-jkr}}{r} q dv + \frac{1}{4\pi} \iint_S \left\{ \frac{\partial \phi}{\partial n} \frac{e^{-jkr}}{r} - \phi \frac{\partial}{\partial n} \left( \frac{e^{-jkr}}{r} \right) \right\} ds$$

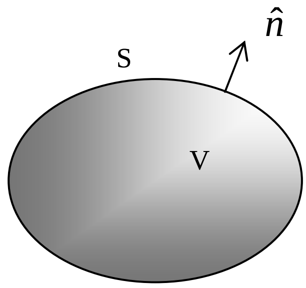
$$\because \int V' \rightarrow 0 \qquad \left( \because \int \frac{1}{r} q 4\pi r^2 dr \rightarrow 0 \ , \ q \leq \mathrm{O}\left( r^{-2} \right) \right)$$

## ***Green's theorem*** （***check2***）

*Green's formula*

$$\iiint_v \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dv = \iint_s \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds$$

$$\nabla^2\psi = \nabla\cdot\nabla\psi$$




［***Proof***］

$$\nabla(\phi\cdot\nabla\psi) = \phi\nabla\cdot\nabla\psi + \nabla\psi\cdot\nabla\phi$$

$$= \phi\nabla^2\psi + \nabla\psi\cdot\nabla\phi$$

$$\therefore \quad \nabla(\phi\cdot\nabla\psi - \psi\cdot\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$$

$$\iiint_V \nabla\{\phi\cdot\nabla\psi - \psi\nabla\phi\}\,dV = \iiint_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dV$$

$$= \oiint_S [\phi\cdot\nabla\psi - \psi\cdot\nabla\phi]\cdot\hat{\mathbf{n}}\;\; dS$$

$$= \oiint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS$$

$\hat{\mathbf{n}}$ : *outward* < *Q.E.D* >

## *The Particular Solution* （*check1*）

*We will indicate that* $\Psi$ *is one of the solutions for* $\nabla^2\psi + k^2\psi = 0$

$\nabla^2\psi + k^2\psi = 0$ の特解を１つ求める。

$$\psi = \frac{e^{-jkr}}{r}$$

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$\frac{\partial r}{\partial x} = \frac{x-x'}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r-(x-x')\frac{x-x'}{r}}{r^2} = \frac{r^2-(x-x')^2}{r^3} = \frac{(y-y')^2+(z-z')^2}{r^3}$$

$$\frac{\partial \psi}{\partial x} = \frac{-jke^{-jkr}r - e^{-jkr}}{r^2}\frac{\partial r}{\partial x} = -\frac{e^{-jkr}}{r^2}[jkr+1]\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} = +\frac{jke^{-jkr}r^2 + e^{-jkr}\cdot 2r}{r^4}\left(\frac{\partial r}{\partial x}\right)^2[jk+1] - \frac{e^{-jkr}}{r^2}jk\left(\frac{\partial r}{\partial x}\right)^2 - \frac{e^{-jkr}}{r^2}[jk+1]\frac{\partial^2 r}{\partial x^2}$$

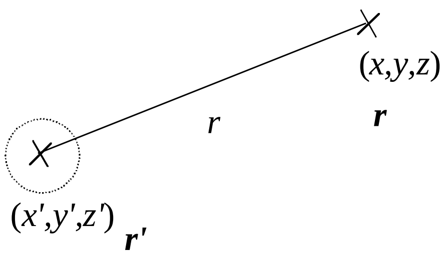
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi = [jk+1]\frac{(jk+2)e^{-jkr}}{r^3}\frac{r^2}{r^2} - \frac{e^{-jkr}}{r^2}jk\frac{r^2}{r^2} - \frac{e^{-jkr}}{r^2}[jk+1]\frac{2r^2}{r^3}$$

$$= \frac{e^{-jkr}}{r}\left[-k^2 + \frac{3jk}{r} + \frac{2}{r^2} - \frac{jk}{r} - \frac{2jk}{r} - \frac{2}{r^2}\right]$$

$$= -k^2\frac{e^{-jkr}}{r} = -k^2\psi$$

$\therefore \quad \nabla^2\psi + k^2\psi = 0$ （ただし、$r \neq 0$ の領域）

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$ ***r*** 系に関する微分




In the solution of vector helmholtz equation, all the integration and derivatives are with respect to the sources $(x',y',z')$. So the roles ***r*** *and* ***r′*** above should be interchanged.

$$\nabla^2 \equiv \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}$$ **r'** 系に関する微分

$\phi=\phi(\mathbf{r}'),\ \psi=\psi(\mathbf{r},\mathbf{r}'),\ q=q(\mathbf{r}')$

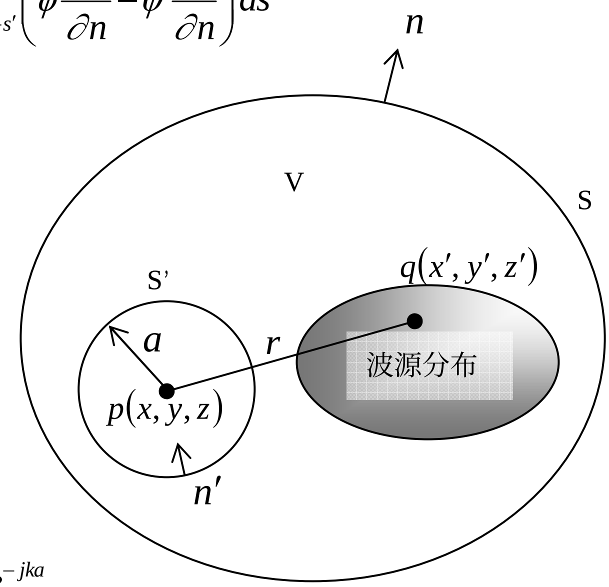
$$\left.\begin{array}{l}\nabla^2\phi+k^2\phi=-q\\ \nabla^2\psi+k^2\psi=0\end{array}\right\}\rightarrow\iint_v\left(\phi\nabla^2\psi-\psi\nabla^2\phi\right)dv=\iint_s\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)ds$$

'系に関する積分
$(x',y',z')$系

$dv$ ↑ $dx'dy'dz'$

$$\iiint_v\left(\phi\left[-k^2\psi\right]-\psi\left[-q-k^2\phi\right]\right)dv=\iint_{s+s'}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)ds$$

$$\iiint_v\psi q dv=\iint_{s+s'}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)ds$$




*On S' with small radius a,*

$$\frac{\partial\psi}{\partial n}=-\frac{\partial\psi}{\partial r}=\frac{jke^{-jkr}r+e^{-jkr}}{r^2}\qquad\left(\psi=\frac{e^{-jka}}{a}\right)$$

$$\Rightarrow\frac{e^{-jkr}}{r^2}=\frac{e^{-jka}}{a^2}\quad(r=a\rightarrow 0)$$

$$\iint_{s'}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)ds\cong\bar{\phi}\iint_{S'}\frac{e^{-jka}}{a^2}ds-\frac{\partial\bar{\phi}}{\partial n}\iint_{S'}\frac{e^{-jka}}{a}ds$$

$\iint ds\Rightarrow 4\pi a^2$ ↓ ↓

$$\rightarrow 4\pi\bar{\phi}-\frac{\partial\Phi}{\partial n}\times\frac{4\pi a^2}{a}\rightarrow 4\pi\bar{\phi}\qquad \bar{\phi}\ \textit{average on } S'$$

$\bar{\phi}$: *the value of* $\phi$ *at* $(x,y,z,)$ *which is the solution* $\bar{\phi}:(x,y,z,)$における$\phi$の値で目的の値

$$\therefore\ \iiint_V\psi(\mathbf{r},\mathbf{r}')q(\mathbf{r}')dv'=\iint_S\left(\phi(\mathbf{r}')\frac{\partial\psi(\mathbf{r},\mathbf{r}')}{\partial n}-\psi(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n}\right)dS'+4\pi\bar{\phi}$$

*Finally we arrive the solution.*

$$\therefore\quad\bar{\phi}\equiv\phi(x,y,z)=\frac{1}{4\pi}\iiint_v\frac{e^{-jkr}}{r}q\,dv+\frac{1}{4\pi}\iint_s\frac{\partial\phi}{\partial n}\frac{e^{-jkr}}{r}-\phi\frac{\partial}{\partial n}\left(\frac{e^{-jkr}}{r}\right)ds$$

$$\int v'\rightarrow 0\qquad\qquad \because\int\frac{1}{r}q4\pi r^2\,dr\rightarrow 0$$

$$q\le O\left(r^{-2}\right)$$

## Linear Differential Equation

（Interpretation of particular and homogeneous solutions）

$$\nabla^2\phi + k^2\phi = -P \quad (\textit{Inhomogeneous equation}) \tag{1}$$

$\phi$ is the solution for inhomogeneous sequation (1).

$\phi_0$ is one of solutions for homogeneous equation $\nabla^2\phi + k^2\phi = 0$ (2).

$\phi + \phi_0$ satisfy equation (1). $\rightarrow$ linear

**Alternative approaches for boundary value problems**

### Approach 1

1) Finding solution which represents contribution from source (Particular solution). The boundary condition is not considered there.
2) Solution for 1) plus general solution for homogeneous eq. satisfy both source singularity and the boundary condition.

### Approach 2

1) Dividing into several source-free regions. Choose general solution for homogeneous eq. combination of solution which satisfies each boundary condition in each region.
2) Determining remaining constants of solution in 1) so that discontinuity (contribution) from source is satisfied.

**Example**

in 1-dimension $\quad P = \delta(x) \qquad \dfrac{d^2}{dx^2}\phi + k^2\phi = -\delta(x)$ (3)

boundary condition $\quad \phi(-b) = 0\,, \quad \phi(a) = 0$

$$\frac{d^2}{dx^2}\phi + k^2\phi = 0 \tag{4}$$

arbitrary choice of two functions from

$$A\sin kx\,,\ B\cos kx\,,\ Ce^{-jkx}\,,\quad De^{jkx} \tag{5}$$

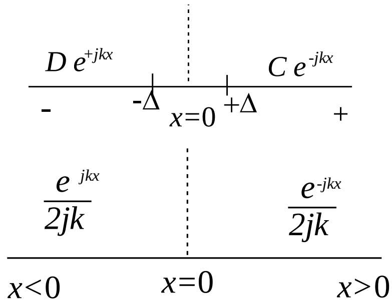
## Approach 1

1) We construct the particular solution by assuming $\phi = De^{jkx} \quad x<0, \ \phi = Ce^{-jkx} \quad x>0$

$$\int_{-\Delta}^{\Delta} dx \rightarrow (3) \qquad \frac{d}{dx}\phi\Big|_{-\Delta}^{\Delta} + \int_{-\Delta}^{\Delta} k^2\phi dx = -1$$

$$-jkC - jkD = -1 \quad (\Delta \rightarrow 0) \ \therefore C + D = \frac{1}{jk}$$

$$continuity \ \ (\phi_+ = \phi_- \rightarrow C = D) \ \therefore C = D = \frac{1}{2jk}$$




2) Particular solution plus general solution for homogeneous eq.(5) must satisfy boundary condition. Arbitrary constants of general solution for homogeneous eq. are determined.

Particular solution $\qquad \phi = e^{jkx}/2jk \quad x<0, \ \phi = e^{-jkx}/2jk \quad x>0$

General solution for homogeneous eq. $\qquad A\sin kx + B\cos kx$

Solution for the boundary value problems.

$$\phi(x<0) = \frac{e^{jkx}}{2jk} + A\sin kx + B\cos kx \tag{7}$$

$$\phi(x>0) = \frac{e^{-jkx}}{2jk} + A\sin kx + B\cos kx \tag{8}$$

Boundary conditions. $\phi(-b) = \frac{e^{-jkb}}{2jk} - A\sin kb + B\cos kb = 0$

$$\phi(a) = \frac{e^{-jka}}{2jk} + A\sin ka + B\cos ka = 0$$

A and B are determined from above two eqs.. Substitution into (7) and (8) gives.

$$\phi = \frac{e^{-jk|x|}}{2jk} - \frac{e^{-jk(x+b)}\sin ka + e^{jk(x-a)}\sin kb}{2jk\sin k(a+b)} \tag{9}$$

## Approach 2

1) Dividing into several source-free regions. Choose general solution for homogeneous eq. combination of solution which satisfies each boundary condition in each region.

$$x<0 \quad \phi(x=-b)=0 \ \therefore -A_1 \sin kb + B_1 \cos kb = 0$$
$$x>0 \quad \phi(x=a)=0 \quad \therefore A_2 \sin ka + B_2 \cos ka = 0$$
$$\therefore B_1 = +A_1 \tan kb$$
$$B_2 = -A_2 \tan kb$$
$$\therefore \textit{general solution for homogeneous eq.}$$
$$\phi = A_1(\sin kx + \tan kb \ \cos kx) \quad x<0$$
$$\phi = A_2(\sin kx - \tan ka \ \cos kx) \quad x>0$$

2) Determining remaining constants of solution in 1) so that discontinuity (contribution) from source is satisfied.

$$\frac{d}{dx}\phi\Big|_{-\Delta}^{\Delta} = kA_2 - kA_1 = -1$$
$$\Delta \to 0$$
$$A_2 - A_1 = -\frac{1}{k} \tag{10}$$

Continuity of $\phi$ at $x = 0$

$$A_1 \tan kb = -A_2 \tan ka \tag{11}$$

from (10) and (11)

$$A_1 = \frac{1}{k}\frac{1}{\left(1+\dfrac{\tan kb}{\tan ka}\right)} = \frac{\tan ka}{k(\tan ka + \tan kb)}$$
$$A_2 = -\frac{1}{k}\frac{1}{\left(1+\dfrac{\tan ka}{\tan kb}\right)} = -\frac{\tan kb}{k(\tan ka + \tan kb)}$$

$$\phi = \begin{cases} \dfrac{\tan ka}{k(\tan ka + \tan kb)}\sin kx + \dfrac{\tan ka \ \tan kb}{k(\tan ka + \tan kb)}\cos kx & x<0 \\ \dfrac{-\tan kb}{k(\tan ka + \tan kb)}\sin kx + \dfrac{\tan ka \ \tan kb}{k(\tan ka + \tan kb)}\cos kx & x>0 \end{cases} \tag{12}$$

## Procedure

- Find one particular solution which is independent of boundary condition.
- Plus general solution for homogeneous eq. so that boundary condition is satisfied.

How to decide coefficient of general solution for homogeneous eq. with out source ?

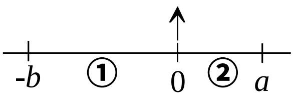
## Alternative choice of the boundaries resulting in the identical solutions

### *Example of 1-dimensional problem* (一次元の問題)

### Original Problem

$$\frac{d^2}{dx^2}\phi(x)+k^2\phi=-q(x)\cong-\delta(x)$$

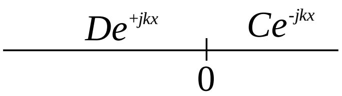
$\phi(a)=\phi(-b)=0 \quad \phi$ *continuity*




斉次解(*Homogeneous equation*)

$$\frac{d^2}{dx^2}\psi+k^2\psi=0 \quad \psi=\alpha e^{-jkx}+\beta e^{+jkx}$$

特殊解(*Particular solution*)



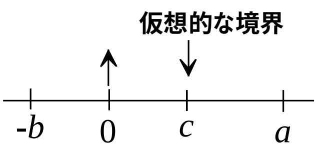

$\int_{-\Delta}^{+\Delta}dx$をする。$\frac{d}{dx}\phi\Big|_{-\Delta}^{+\Delta}+k^2\int_{-\Delta}^{+\Delta}\phi dx=-\int_{-\Delta}^{+\Delta}\delta(x)dx=-1$

$$-jkC-jkD=-1$$

$$\phi\equiv\begin{cases}\alpha e^{-jkx}+\beta e^{+jkx}+De^{+jkx} & x<0\\ \alpha e^{-jkx}+\beta e^{+jkx}+Ce^{-jkx} & x>0\end{cases}$$

*from continuity condition at* $x=0 \quad D=C,\ \phi(a)=\phi(-b)=0$

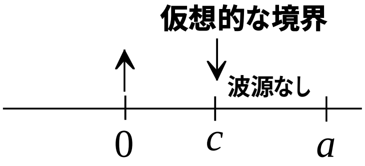
$$\phi=\frac{e^{-jk|x|}}{2jk}-\frac{\sin ka\, e^{-jk(x+b)}+\sin kb\, e^{+jk(x+a)}}{2jk\sin k(a+b)} \qquad \text{(A)}$$




ここで$\phi(c)$を計算すると　$\phi$ *reads as*

$$\phi(c)=\frac{e^{-jkc}}{2jk}-\frac{\sin ka\ \ e^{-jk(c+b)}+\sin kb\ \ e^{+jk(c+a)}}{2jk\sin k(a+b)}=T \qquad \text{(B)}$$

## *Equivalent problems* (**等価な**新しい問題)




$$\nabla^2\phi + k^2\phi = 0$$
$$\phi(a) = 0, \quad \phi(c) = T$$

$$\phi = \alpha e^{-jka} + \beta e^{+jkx}$$

$$\left.\begin{aligned}\phi(a) &= \alpha e^{-jka} + \beta e^{+jka} = 0\\ \phi(c) &= \alpha e^{-jkc} + \beta e^{+jkc} = T\end{aligned}\right\} \quad \alpha\left(e^{-jka+jkc} - e^{+jka-jkc}\right) = -e^{jka}T$$

$$\alpha = \frac{-e^{jka}}{2j\sin(kc-ka)}T \qquad \beta = \frac{e^{-jka}}{2j\sin(kc-ka)}T$$

$$T = \frac{1}{2jk}\left(e^{-jkc} - \frac{\sin ka \quad e^{-jk(c+b)} + \sin kb \quad e^{+jk(c-a)}}{\sin k(a+b)}\right) \qquad \text{(C)}$$

$$= \frac{1}{2jk\sin k(a+b)}\left(\sin kb \quad e^{-jkc}e^{+jka} - \sin kb \quad e^{-jka}e^{+jkc}\right)$$

$$= \frac{\sin kb}{2jk\sin k(a+b)} 2j\sin k(a-c)$$

$$T = \frac{\sin kb \quad \sin k(a-c)}{k\sin k(a+b)}$$

$$\alpha = \frac{-\sin kb \quad e^{+jka}}{2jk\sin k(a+b)} \qquad \beta = \frac{\sin kb \quad e^{-jka}}{2jk\sin k(a+b)}$$

予想通り、C の値に依存しない。

Independent of the position C, as is expected.

$$\therefore \phi = \frac{\sin kb}{2jk\sin k(a+b)}\left(-e^{jka}e^{-jkx} + e^{-jka}e^{+jkx}\right) \qquad \text{(D)}$$

Since x > 0 （*A*） -> （E） ***Identical!!*** ↕ 等しい

$$\phi = \frac{1}{2jk\sin k(a+b)}e^{-jkx}\left(\sin k(a+b) - \sin ka\, e^{-jkb}\right) - \frac{\sin kb}{2jk\sin k(a+b)}e^{+jkx}e^{-jka} \qquad \text{(E)}$$

∴境界値 *ϕ(c)* は真の波源と同じ働きをする。

The boundary value *(c)* acts as if it was the **true source.**

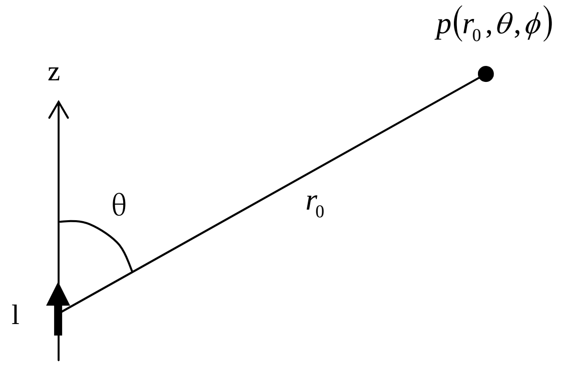
## A small dipole 微少ダイポール （**Example** 計算例）

$$\mathbf{A} = \frac{\mu}{4\pi}\int_v \mathbf{J}\frac{e^{-jkr}}{r}dv$$

*Free space* 自由空間

$$\mathbf{J} = \hat{z}I\delta(x)\delta(y)f(z)$$

$$f(z) = \begin{cases} 1 & (0 \sim l) \\ 0 & (otherwise) \end{cases}$$




$$dv = dxdydz$$

$$\mathbf{A} = \frac{\mu}{4\pi}\frac{e^{-jkr_0}}{r_0}\hat{z}I\int_0^l \delta(x)\delta(y)f(z)dxdydz$$
$$= \frac{\mu Il}{4\pi}\frac{e^{-jkr_0}}{r_0}\hat{z}$$

Electric fields are calculated in terms of ***A***.
これを用いて、電磁界は次式で計算

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\varepsilon\mu}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$$

***Spherical coordinate*** 極座標

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$$

$$\mu H_r = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right\}$$

$$\mu H_\theta = \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right\}$$

$$\mu H_\phi = \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right\}$$

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$$\mathbf{A} = \frac{\mu I l e^{-jkr}}{4\pi r}\hat{z}$$

$$= \frac{\mu I l e^{-jkr}}{4\pi r}(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

$$\frac{\partial}{\partial r}\left(rA_\theta\right) = \frac{\partial}{\partial r}\left(-\frac{\mu I l e^{-jkr}}{4\pi}\sin\theta\right) = -\frac{\mu I l}{4\pi}\sin\theta\frac{\partial}{\partial r}(e^{-jkr}) = jk\frac{\mu I l}{4\pi}e^{-jkr}\sin\theta$$

$$\frac{\partial A_r}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{\mu I l e^{-jkr}}{4\pi r}\cos\theta\right) = \frac{\mu I l e^{-jkr}}{4\pi r}\frac{\partial}{\partial\theta}\left(\cos\theta\right) = -\frac{\mu I l e^{-jkr}}{4\pi r}\sin\theta$$

−−−−−−−−−−−−−−−−−

$$H_\phi = \frac{1}{\mu r}\left(jk + \frac{1}{r}\right)\frac{\mu I l}{4\pi}e^{-jkr}\sin\theta$$

$$= \frac{jkIl}{4\pi r}\sin\theta\left(1 + \frac{1}{jkr}\right)e^{-jkr}$$

$$= \frac{jIl}{2\lambda r}e^{-jkr}\sin\theta\left(1 + \frac{1}{jkr}\right)$$

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\mu}$$

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$$\nabla\cdot\mathbf{A} = \frac{\partial}{\partial z}A_z \qquad , \ A_z = \frac{\mu Il}{4\pi r}e^{-jkr}$$

$$\nabla(\nabla\cdot\mathbf{A}) = \left(\frac{\partial}{\partial r}\left(\frac{\partial A_z}{\partial z}\right), \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial A_z}{\partial z}\right), \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_z}{\partial z}\right)\right)$$

$\Downarrow$

0 軸対称: *Axial symmetry*

$$\frac{\partial A_z}{\partial z} = \frac{\mu Il}{4\pi}\left(-\frac{e^{-jkr}}{r^2}(jkr+1)\frac{z}{r}\right)$$

$$= \frac{\mu Il}{4\pi}\frac{-(jkr+1)}{r^2}e^{-jkr}\cos\theta$$

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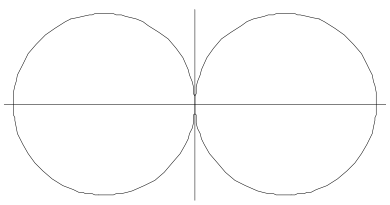
$$\mathbf{E} = \hat{r}\frac{\eta Il}{2\pi r^2}e^{-jkr}\left(1+\frac{1}{jkr}\right)\cos\theta + \hat{\theta}j\frac{\eta Il}{2\lambda r}e^{-jkr}\left(1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right)\sin\theta + \hat{\phi}0$$

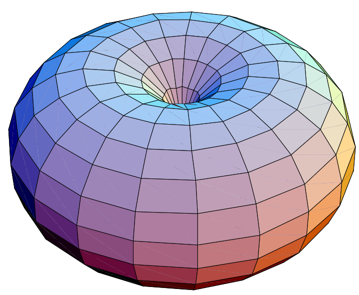
$\Downarrow$

放射項: *Radiation term*

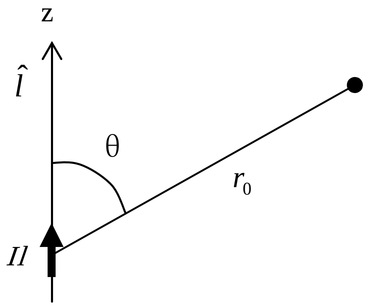
$kr >> 1$

$$\left.\begin{aligned}\mathbf{E} &= \hat{\theta}j\frac{\eta Il}{2\lambda r}e^{-jkr}\sin\theta \\ \mathbf{H} &= \hat{\phi}j\frac{Il}{2\lambda r}e^{-jkr}\sin\theta\end{aligned}\right) \quad E_\theta = \eta H_\phi, \ \eta = \sqrt{\frac{\mu}{\varepsilon}}, \quad k = \frac{2\pi}{\lambda}$$





# Interpretation of each term in the fields




$$\mathbf{H}=\left\{\underline{\frac{jIl}{2\lambda r}e^{-jkr}\sin\theta}+\underline{\frac{Il}{4\pi r^2}e^{-jkr}\sin\theta}\right\}\hat{\phi}$$

| 放射界 (Radiation) | 誘導界 (Inductive) | ビオサバール (Biot Savart) |
|---|---|---|

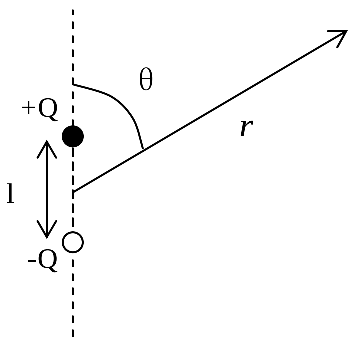
$$dH=\frac{Idl\times\hat{r}}{4\pi r}$$

↓ ↓ ↓

$$\mathbf{E}=\underline{j\frac{\eta Il}{2\lambda r}e^{-jkr}\sin\theta\hat{\theta}}+\underline{\frac{\eta Il}{2\pi r^2}e^{-jkr}\cos\theta\hat{r}}+\underline{\frac{\eta Il}{4\pi r^2}e^{-jkr}\sin\theta\hat{\theta}}$$

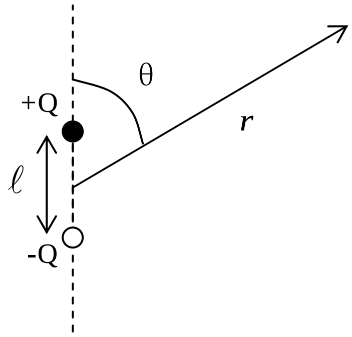
$$\left.\begin{array}{l}+\dfrac{\left(\frac{Il}{j\omega}\right)}{2\pi\varepsilon r^3}\cos\theta e^{-jkr}\hat{r}\\ +\dfrac{\left(\frac{Il}{j\omega}\right)}{4\pi\varepsilon r^3}\sin\theta e^{-jkr}\hat{\theta}\end{array}\right)$$

準静電界（$Quasi\text{-}Static$） $\dfrac{Il}{j\omega}=Ql$ :ダイポール・モーメント




$$\frac{d}{dt}Q=I \quad , \quad j\omega Q=I \quad , \quad Il=j\omega lQ$$

## Relations between the dipole and the quasi-static terms.
(ダイポールモーメントと準静電界項の対応)



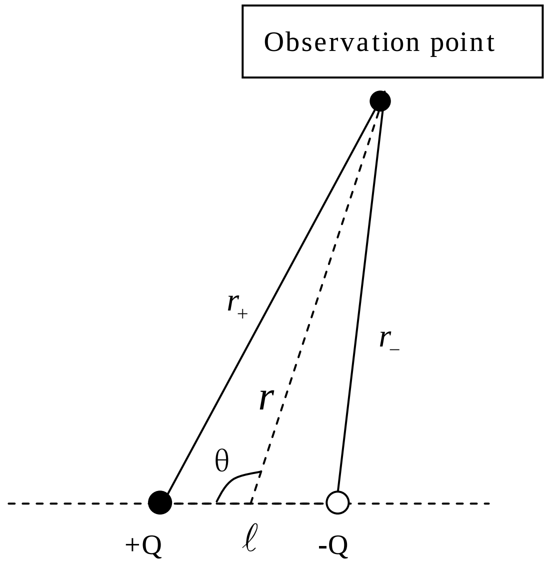

$$\frac{d}{dt}Q = I \quad , \quad j\omega Q = I \quad , \quad I\ell = j\omega\ell Q$$

$$\mathbf{E} = -\nabla\Psi = -\left(\frac{\partial\Psi}{\partial r}, \frac{1}{r}\frac{\partial\Psi}{\partial\theta}, \frac{1}{r\sin\phi}\frac{\partial\Psi}{\partial\phi}\right)$$

$$\Psi = \frac{Q\ell}{4\pi\varepsilon}\frac{\cos\theta}{r^2}$$ （後述）

$$\mathbf{E} = -\left(\frac{Q\ell}{4\pi\varepsilon}\right)\left(-2\frac{\cos\theta}{r^3}, -\frac{\sin\theta}{r^3}, 0\right)$$ $r\ l << r$

$$= \frac{Q\ell\cos\theta}{2\pi\varepsilon r^3}\hat{r} + \frac{Q\ell\sin\theta}{4\pi\varepsilon r^3}\hat{\theta}$$ $\rightarrow$ 前頁の準静電界




$$\Psi_+ = \frac{Q}{4\pi\varepsilon r_+}\ \Psi_- = -\frac{Q}{4\pi\varepsilon r_-}$$

$$r_+ = r + \frac{l}{2}\sin\left(\theta - \frac{\pi}{2}\right) = r - \frac{l}{2}\cos\theta$$

$$r_- = r - \frac{l}{2}\sin\left(\theta - \frac{\pi}{2}\right) = r + \frac{l}{2}\sin\theta$$

$$\Psi = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = \frac{Q}{4\pi\varepsilon}\frac{r_- - r_+}{r_+ r_-}$$

$$= \frac{Q}{4\pi\varepsilon}\frac{l\cos\theta}{r^2} = \frac{Ql}{4\pi\varepsilon}\frac{\cos\theta}{r^2}$$ $< Q.E.D >$