Random Process
(確率過程)

通信伝送工学

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Mathematical Model

Deterministic models（決定論）

- The solution of a set of mathematical equations
- The exact outcomes of the experiment

Ex) Circuit theory, electromagnetic theory

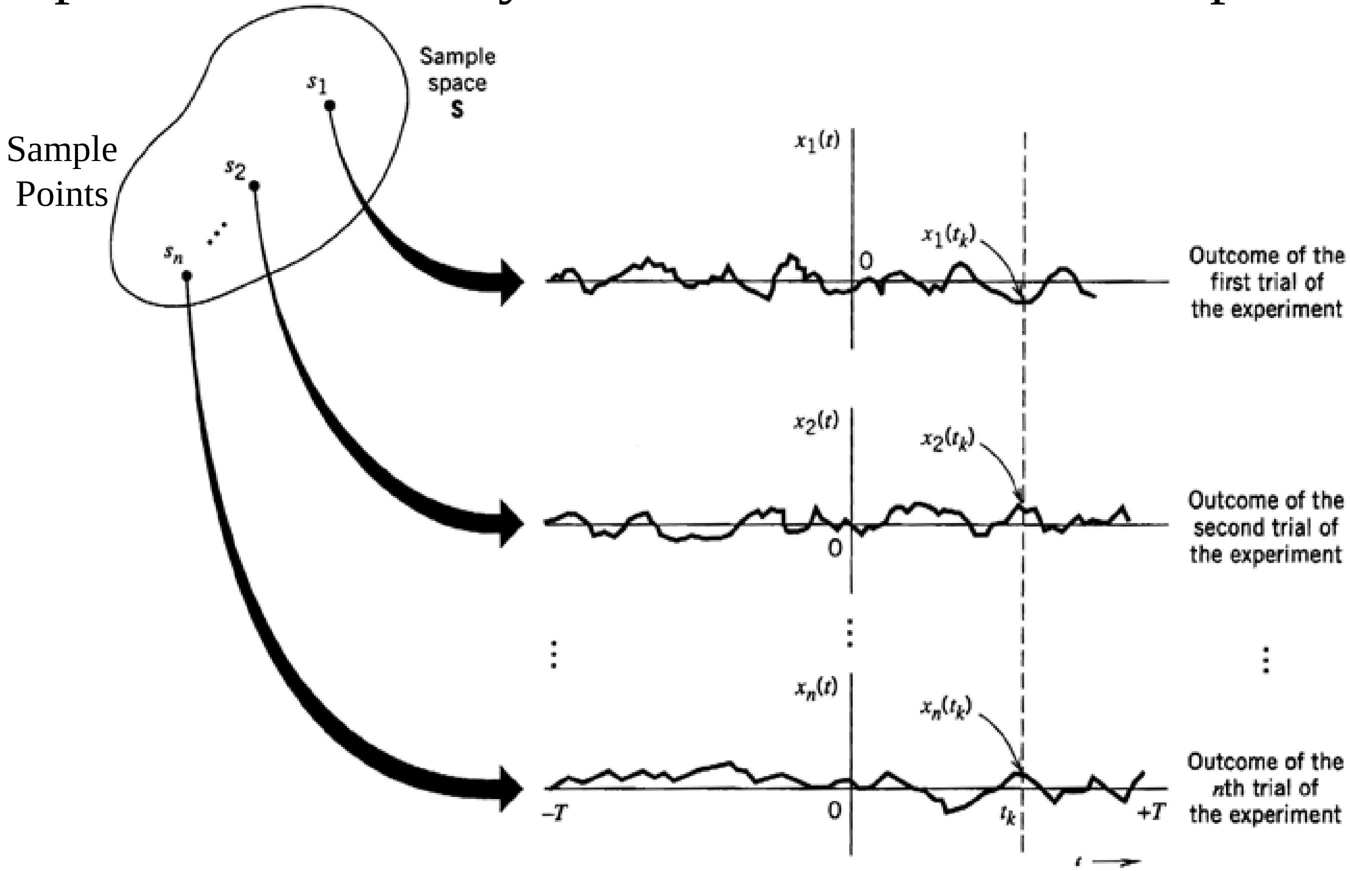
Stochastic (Random) models（確率論）

- Not possible to predict the exact value in advance
- Described by statistical parameters (Average power, po wer spectral density）

Ex) Thermal Noise in Comm. Systems (random motion of Electrons)

Random Process

- Random process is function of time
- Impossible to exactly define the value before experiment



Sample space or ensemble composed of functions of time

⇒ Random process or Stochastic process

Random Process(2)

- Assign to sample point *s* a function of time

$$X(t,s), \qquad -T \le t \le T$$

- For a fixed sample point s_j,

$$x_j(t) = X(t, s_j) \qquad \text{: Sample function}$$

- For a fixed time instance t_k,

$$\{x_1(t_k), x_2(t_k), \ldots, x_n(t_k)\} = \{X(t_k, s_1), X(t_k, s_2), \ldots, X(t_k, s_n)\}$$

$\Rightarrow$ Random variable

- Random process : *X(t)* is simply used

Stationary Process

- Process is called "stationary" when

$$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k)$$

$\Rightarrow$ Strictly stationary

- First order stationary

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x)$$

- Second order stationary

$$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(0),X(t_2-t_1)}(x_1,x_2)$$

Mean, Correlation, Covariance

- Mean of the process X(t)

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x \cdot f_{X(t)}(x)dx$$

$$\mu_X(t) = \mu_X$$ ← First order stationary

- Autocorrelation between $X(t_1)$ and $X(t_2)$

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)dx_1 dx_2$$

$$R_X(t_1, t_2) = R_X(t_2 - t_1)$$ ← Second order stationary

- Auto covariance function

$$C_X(t_1, t_2) = E[(X(t_1) - \mu_X)(X(t_2) - \mu_X)]$$

$$= R_X(t_2 - t_1) - \mu_X^2$$

Wide-sense Stationary (WSS)

- ***Wide-sense*** stationary (weakly stationary) : simply "***stationary***"

$$\mu_X(t) = \mu_X$$

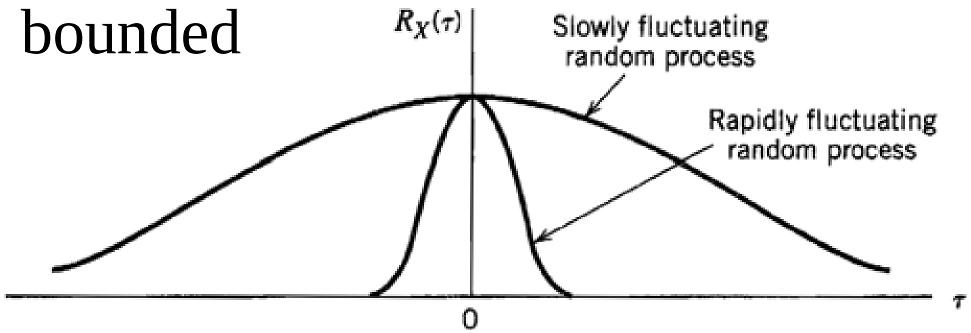
$$R_X(t_1, t_2) = R_X(t_2 - t_1)$$

- Properties of Autocorrelation function $R_X(\tau) = E\left[X(t+\tau)X(t)\right]$

$$R_X(0) = E\left[X^2(t)\right]$$: Mean-square value

$$R_X(\tau) = R_X(-\tau)$$: even function

$$\left|R_X(\tau)\right| \le R_X(0)$$: bounded



Cross-Correlation Function

- Cross correlations between two random process of X(t) and Y(t)

$$R_{XY}(t,u) = E[X(t)Y(u)]$$
$$R_{YX}(t,u) = E[Y(t)X(u)]$$

- Correlation matrix

$$\boldsymbol{R}(t,u) = \begin{bmatrix} R_X(t,u) & R_{XY}(t,u) \\ R_{YX}(t,u) & R_Y(t,u) \end{bmatrix}$$

$$\boldsymbol{R}(\tau) = \begin{bmatrix} R_X(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_Y(\tau) \end{bmatrix} \qquad \because \tau = t - u$$

$\Rightarrow$ jointly stationary

- Symmetry property

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

Ergodic Process

- **Expectation** (or ensemble average) E[X(t)] means averages ***across*** the process
- **Time average** means averages ***along*** the process

$$\mu_x(T) = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

x(t) : sample function of X(t)

$$E[\mu_x(T)] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]dt$$

$$= \frac{1}{2T}\int_{-T}^{T} \mu_X dt = \mu_X$$

μ_X : ensemble average of X(t)

- Time average represents an unbiased estimate of ensemble-averaged mean μ_X

Ergodic Process (2)

- Ergodic in the mean if the following are satisfied:

$$\lim_{T\to\infty} \mu_x(T) = \mu_X$$

$$\lim_{T\to\infty} \operatorname{var}\left[\mu_x(T)\right] = 0$$

- Ergodic in the autocorrelation function if the following are satisfied:

$$R_x(\tau,T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)dt$$

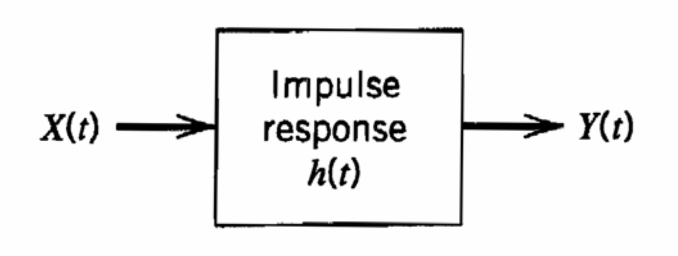
$$\lim_{T\to\infty} R_x(\tau,T) = R_X(\tau)$$

$$\lim_{T\to\infty} \operatorname{var}\left[R_x(\tau,T)\right] = 0$$

- For a random process to be ergodic, it has to be stationary

Linear Time-Invariant Filter

- Suppose a random process X(t) is applied as input to LTI filter h(t), producing a random process Y(t)



$$Y(t) = \int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1) d\tau_1$$

$$\mu_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1) d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1) E[X(t - \tau_1)] d\tau_1 = \int_{-\infty}^{\infty} h(\tau_1) \mu_X(t - \tau_1) d\tau_1$$

$$\mu_Y(t) = \mu_X \int_{-\infty}^{\infty} h(\tau_1) d\tau_1 = \mu_X H(0)$$

Linear Time-Invariant Filter(2)

- Consider autocorrelation function of Y(t)

$$R_Y(t,u) = E\left[Y(t)Y(u)\right]$$
$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(u-\tau_2)d\tau_2\right]$$

$$R_Y(t,u) = \int_{-\infty}^{\infty} d\tau_1 h(\tau_1)\int_{-\infty}^{\infty} d\tau_2 h(\tau_2)E\left[X(t-\tau_1)X(u-\tau_2)\right]$$
$$= \int_{-\infty}^{\infty} d\tau_1 h(\tau_1)\int_{-\infty}^{\infty} d\tau_2 h(\tau_2)R_X(t-\tau_1, u-\tau_2) \qquad \because \tau = t-u$$

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)d\tau_1 d\tau_2$$

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)d\tau_1 d\tau_2$$ Mean square value

Power Spectral Density

- Random process in linear systems in ***frequency domain***

- Frequency response of the system

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f)\exp(j2\pi f\tau_1)df \qquad \text{Inverse Fourier Transform}$$

- Mean-square value of output process Y(t)

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(f)\exp(j2\pi f\tau_1)df\right]h(\tau_2)R_X(\tau_2-\tau_1)d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty} df H(f)\int_{-\infty}^{\infty} d\tau_2 h(\tau_2)\int_{-\infty}^{\infty} R_X(\tau_2-\tau_1)\exp(j2\pi f\tau_1)d\tau_1$$

Power Spectral Density (2)

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} df H(f) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2) \exp(j2\pi f \tau_2)$$

$$\cdot \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau) d\tau$$

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} df \left|H(f)\right|^2 \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau) d\tau$$

Power spectrum density of stationary process X(t)

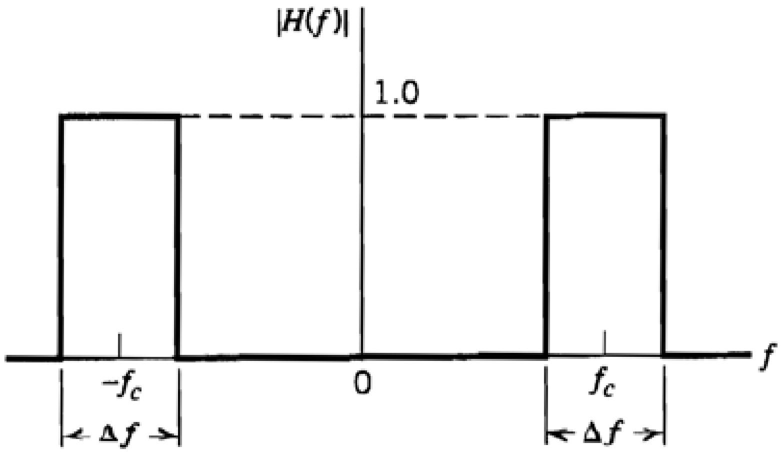
$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau) d\tau$$

$$\therefore E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} \left|H(f)\right|^2 S_X(f) df$$

Power Spectral Density (3)

- Ideal bandpass filter case

$$|H(f)| = \begin{cases} 1, & |f \pm f_c| < \frac{1}{2}\Delta f \\ 0, & |f \pm f_c| > \frac{1}{2}\Delta f \end{cases}$$



$$\Delta f << f_c$$

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df$$

$$\approx (2\Delta f) S_X(f_c)$$

Properties of PSD

- ***Einstein-Wiener-Khintchine*** relation

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)df$$

If either the autocorrelation function or power spectral density is known, the other can be found exactly

- Property 1: DC value of PSD = total area of autocorrelation function

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)d\tau$$

- Property 2: mean-square value of the process = total area of PSD

$$E\left[X^2(t)\right] = \int_{-\infty}^{\infty} S_X(f)df$$

Properties of PSD (2)

- Property 3: PSD is always nonnegative

$$S_X(f) \geq 0 \quad (\forall f)$$

- Property 4: PSD is even function for real-valued random process

$$S_X(-f) = S_X(f)$$

- Property 5: Normalized PSD can be associated with a probability density function

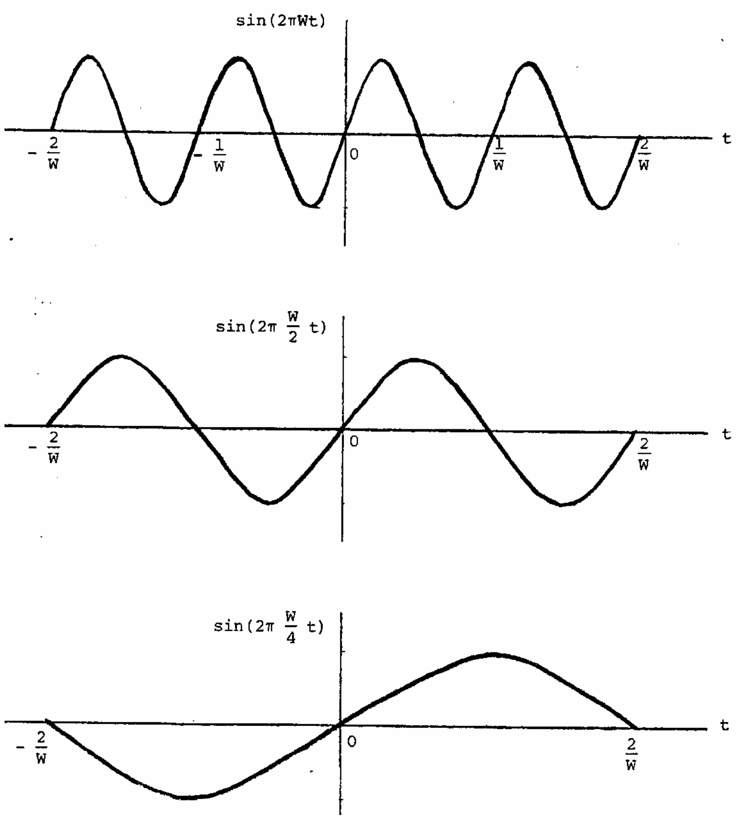
$$p_X(f) = \frac{S_X(f)}{\int_{-\infty}^{\infty} S_X(f)df}$$

Example (1)

確立過程X(t)は次のように定義される．

$$X(t) = \sin(2\pi f_c t)$$

ここで，fc は[0, W]に一様分布 (uniform distribution) する確率変数である．
X(t)が定常ではない(Nonstationary)理由を説明せよ．
* ヒント： f=W/4, W/2, Wの場合に確率過程X(t)のサンプル関数について調べよ．



$$f_{X(t)}(x) = f_{X(t+\tau)}(x) = F_X(x)$$

$$f_{X(-W/4)}(x) \neq f_{X(W/4)}(x)$$

Example (2)

確立過程X(t)は次のように定義される．

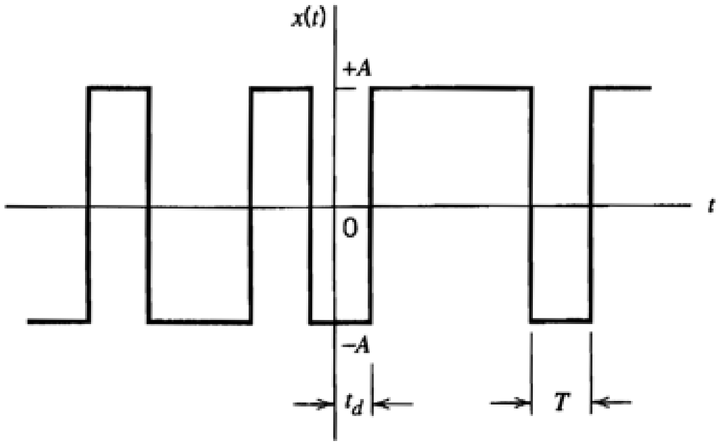
$$X(t) = A\cos(2\pi f_c t + \Theta)$$

ここで，A，fcは定数，Θ は[-π, π]に一様分布 (uniform distribution) する確率変数である．$R_X(\tau)$ を求めよ．

Example (3)

Random binary波形
$R_X(\tau)$ を求めよ.

$$f_{T_d}(t_d) = \begin{cases} 1/T & 0 \le t_d \le T \\ 0 & otherwise \end{cases}$$



Example (4)

確率過程X(t)は

$$X(t) = A\cos(2\pi f_c t)$$

ここで，Aは平均がゼロ，分散が σ_A^2 のガウス確率変数である．この確率過程は理想積分器に入力し，その出力は次式のようである．

$$Y(t) = \int_0^t X(\tau)d\tau$$

(a) ある時刻t_k における出力Y(t)の確率密度関数を求めよ．
(b) Y(t)は，定常(stationary)であるかどうか説明せよ．
(c) Y(t)は，エルゴード性 (Ergothic) かどうか説明せよ．

Example (5)

確立過程X(t)の自己相関関数 における次の性質を証明せよ.

(a) X(t)がDC(直流)成分 を含む場合, $R_X(\tau)$ はA^2の定数成分を含む.

(b) X(t)が正弦波成分を含む場合, $R_X(\tau)$は同じ周波数の正弦波成分を含む.