Integral Expression of EM Fields （Summary）

In terms of tangential and normal component of *E* and *H* on S as

$$\mathbf{E}=\frac{1}{4\pi}\int\left(-j\omega\mu\phi\,\mathbf{J}-\mathbf{J}^{*}\times\nabla\phi+\nabla\phi\,\frac{\rho}{\varepsilon}\right)dv+\frac{1}{4\pi}\oint\left\{-j\omega\mu\phi\left(\mathbf{H}\times\hat{\mathbf{n}}\right)-\left(\hat{\mathbf{n}}\times\mathbf{E}\right)\times\nabla\phi-\nabla\phi\left(\hat{\mathbf{n}}\cdot\mathbf{E}\right)\right\}ds$$

$$\mathbf{H}=\frac{1}{4\pi}\int_{v}\left(-j\omega\varepsilon\,\mathbf{J}^{*}\phi+\mathbf{J}\times\nabla\phi+\nabla\phi\,\frac{\rho^{*}}{\mu}\right)dv+\frac{1}{4\pi}\oint_{s}\left\{-j\omega\varepsilon\phi\left(\hat{\mathbf{n}}\times\mathbf{E}\right)-\left(\hat{\mathbf{n}}\times\mathbf{H}\right)\times\nabla\phi-\nabla\phi\left(\hat{\mathbf{n}}\cdot\mathbf{H}\right)\right\}ds$$

Alternatively, in terms of tangential components only as

$$E=\frac{1}{4\pi}\int_{v}\left(\overset{①}{-j\omega\mu\phi J}-\overset{③}{J^{*}\times\nabla\phi}+\overset{②}{\nabla\phi\frac{\rho}{\varepsilon}}\right)dv+\frac{1}{4\pi}\int_{s}\left\{\overset{①}{-j\omega\mu\phi\left(H_{-}\times n\right)}-\overset{③}{\left(n\times E\right)\times\nabla\phi}+\overset{②}{\frac{1}{j\omega\varepsilon}\left[\left(H_{-}\times n\right)\cdot\nabla\right]\nabla\phi}\right\}ds$$

$$H=\frac{1}{4\pi}\int_{v}\left(\overset{⑥}{-j\omega\varepsilon\phi J^{*}}+\overset{④}{J\times\nabla\phi}+\overset{⑤}{\nabla\phi\frac{\rho^{*}}{\mu}}\right)dv+\frac{1}{4\pi}\int_{s}\left\{\overset{⑥}{-j\omega\varepsilon\phi\left(n\times E_{-}\right)}-\overset{④}{\left(n\times H\right)\times\nabla\phi}+\overset{⑤}{\frac{1}{j\omega\mu}\left[\left(n\times E_{-}\right)\cdot\nabla\right]\nabla\phi}\right\}ds$$

番号の対応は 5-2 ページ

We can conclude that fields can be uniquely determined by the Sources inside V and on S only.

$$\left.\begin{array}{ll}v\text{内の} & J,J^{*}\\ s\text{上の} & N\times E,n\times H\end{array}\right\}\ \text{のみで決定される。}$$

ds , dv , V は全て積分座標系。

$J_s\equiv H_{-}\times n$ $\quad\quad$ $\rho_s^{*}=-\mu n\cdot H$

n は外向き

$J_s^{*}\equiv n\times E_{-}$ $\quad\quad$ $\rho=-\varepsilon n\cdot E$

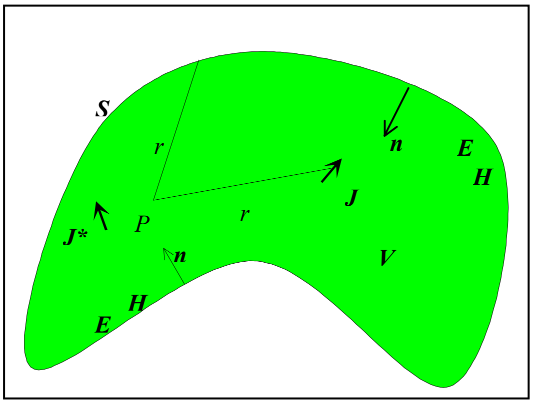
It seems that one has to calculate 6 terms by integration independently. Practical calculation, on the contrarty, is rather straightforward. Once two types of potentials, that is for electric line current and for magnetic line current, are obtained, six terms can be derived systematically, referring to the formula using vector potentials under Lorentz condition.

How to calculate the fields from engineering point of view??

Two types of sources. ($\boldsymbol{n}$: normal into inside V)

1) Electric currents $\boldsymbol{J}$ in V and equivalent surface electric currents $\boldsymbol{J}_S = \boldsymbol{n}\times\boldsymbol{H}$ on S.

2) Magnetic currents (non-physical) $\boldsymbol{J^*}$ in V and equivalent surface magnetic currents $\boldsymbol{J}_S{}^* = \boldsymbol{E}\times\boldsymbol{n}$ on S.



3) Vector potentials

$$\boldsymbol{A} = \frac{\mu}{4\pi}\oint_{S+V}(\boldsymbol{J}+\boldsymbol{J}_S)\phi dv, \qquad \boldsymbol{B} = \frac{\varepsilon}{4\pi}\oint_{S+V}(\boldsymbol{J}^*+\boldsymbol{J}_S{}^*)\phi dv$$ are calculated.

Volume and surface integrations.

4) With reference to the vector field derived in terms of potentials (on the next page), duality is utilized and six terms are calculated by two potentials $\boldsymbol{A}$ and $\boldsymbol{B}$ as well as their simple derivatives.

$$E = \frac{1}{4\pi}\int_s\left\{-j\omega\mu\phi J_s - J_s^*\times\nabla\phi + \nabla\phi\frac{\rho_s}{\varepsilon}\right\}dv$$

$$H = \frac{1}{4\pi}\int_s\left\{-j\omega\varepsilon\phi J_s^* + J_s\times\nabla\phi + \nabla\phi\frac{\rho_s^*}{\mu}\right\}dv$$

$$J_s = n_i\times H \qquad \rho_s^* = n_i\cdot H\mu \qquad A_s = \frac{\mu}{4\pi}\oint J_s\phi ds$$

$$J_s^* = E\times n_i \qquad \rho_s = \varepsilon\, n_i\cdot E \qquad B_s = \frac{\varepsilon}{4\pi}\oint J_s^*\phi ds$$

$$E = -j\omega A_s + \frac{\nabla\nabla\cdot A_s}{j\omega\varepsilon\mu} - \frac{\nabla\times B_s}{\varepsilon}, H = \frac{\nabla\times A_s}{\mu} + \frac{\nabla\nabla\cdot B_s}{j\omega\varepsilon\mu} - j\omega B_s$$

*1 *2 *3 *4 *5 *6 ← 番号の対応

5) If the observer is in the far field of the source and the surface, further simplification applies which dispenses with derivatives.

From electric currents

$$\mathbf{E}_A = -\mathbf{j\omega}\left(\overset{①}{\mathbf{A}} - \hat{\mathbf{r}}\left(\overset{②}{\mathbf{A}\cdot\hat{\mathbf{r}}}\right)\right) \qquad \cdots \quad *1, \quad *2$$

$$\mathbf{H}_A = \frac{1}{\eta}\left(\overset{③}{\hat{\mathbf{r}}\times\mathbf{E}_A}\right) \qquad \eta = \sqrt{\mu/\varepsilon} \approx 120\pi \qquad \cdots \quad *4$$

From magnetic currents

$$\mathbf{H}_B = -\mathbf{j\omega}\left(\overset{⑥}{\mathbf{B}} - \hat{\mathbf{r}}\left(\overset{⑤}{\mathbf{B}\cdot\hat{\mathbf{r}}}\right)\right) \qquad \cdots \quad *5, \quad *6$$

$$\mathbf{E}_B = -\eta\left(\overset{④}{\hat{\mathbf{r}}\times\mathbf{H}_B}\right) \qquad \eta = \sqrt{\mu/\varepsilon} \approx 120\pi \qquad \cdots \quad *3$$

Finally, we get

$$\mathbf{E} = -\mathbf{j\omega}\left(\overset{①}{\mathbf{A}} - \hat{\mathbf{r}}\left(\overset{②}{\mathbf{A}\cdot\hat{\mathbf{r}}}\right)\right) - \eta\left(\overset{④}{\hat{\mathbf{r}}\times(-j\omega\mathbf{B})}\right) \qquad \cdots \quad *1, \quad *2, \quad *3$$

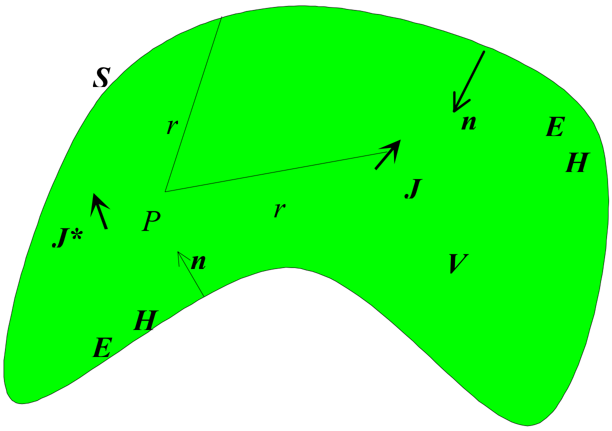
$$\mathbf{H} = \frac{1}{\eta}\left(\overset{③}{\hat{\mathbf{r}}\times(-j\omega\mathbf{A})}\right) - \mathbf{j\omega}\left(\overset{⑥}{\mathbf{B}} - \hat{\mathbf{r}}\left(\overset{⑤}{\mathbf{B}\cdot\hat{\mathbf{r}}}\right)\right) \qquad \cdots \quad *4, \quad *5, \quad *6$$

Explicitly, fields are given by

$$\boldsymbol{E} = -j\omega(A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\varphi}}) - j\omega\eta(-B_\theta\hat{\boldsymbol{\varphi}} + B_\phi\hat{\boldsymbol{\theta}})$$

$$\boldsymbol{H} = -j\omega(B_\theta\hat{\boldsymbol{\theta}} + B_\phi\hat{\boldsymbol{\varphi}}) - j\omega\, {}^{1}\!/_{\eta}(A_\theta\hat{\boldsymbol{\varphi}} - A_\phi\hat{\boldsymbol{\theta}})$$

References 参考:



$$\nabla \times H = j\omega\varepsilon E + J$$
$$\nabla \times E = -j\omega\mu H$$

Vector potential

$$\mu H \equiv \nabla \times A$$

Vector Helmholtz Equation

$$\nabla^2 A + k^2 A = -\mu J$$

Free space Green's Function

$$A = \frac{\mu}{4\pi}\int_v J\,\frac{e^{-jkr}}{r}\,dv$$

Fields are expressed in terms of vector potential **A** as:

$$H = \frac{1}{\mu}\nabla \times A$$

$$E = -j\omega A + \underbrace{\frac{\nabla(\nabla\cdot A)}{j\omega\varepsilon\mu}}_{-\nabla\phi}$$

When only the far fields are considered, we have a TEM wave.

$|r| \to \infty$ の中は簡単になり　$E = -j\omega\left(A - \hat{r}\left(A\cdot\hat{r}\right)\right)$

TEM波　$H = \frac{1}{\eta}\left(\hat{r}\times E\right)$　$\eta = \sqrt{\mu/\varepsilon} \approx 120\pi$

◆ *Duality in Maxwell's Equations*

$$\nabla \times E = -j\omega\mu H - J^*$$
$$\nabla \times H = \quad j\omega\varepsilon E + J$$

$\boldsymbol{E} \to \boldsymbol{H}$　$\varepsilon \leftrightarrow \mu$

$\boldsymbol{H} \to -\boldsymbol{E}$

$\boldsymbol{J}^* \to -\boldsymbol{J}$

$\boldsymbol{J} \to \boldsymbol{J}^*$

$\rho \to \rho^*$　$\rho^* \to -\rho$

Free space Green's Function

$$\mathbf{B} = \frac{\boldsymbol{\mu}}{4\boldsymbol{\pi}}\int_v \mathbf{J}^*\,\frac{\mathbf{e}^{-\mathbf{jkr}}}{\mathbf{r}}\,\mathbf{dv}$$

Fields are expressed in terms of vector potential A as:

$$\mathbf{E} = \frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\nabla \times \mathbf{B}$$

$$\mathbf{E} = -\mathbf{j}\boldsymbol{\omega}\mathbf{B} + \underline{\frac{\nabla(\nabla\cdot\mathbf{B})}{\mathbf{j}\boldsymbol{\omega\mu\varepsilon}}}$$

When only the far fields are considered, we have a TEM wave.

$$\mathbf{H} = -\mathbf{j}\boldsymbol{\omega}\left(\mathbf{B} - \hat{\mathbf{r}}\left(\mathbf{B}\cdot\hat{\mathbf{r}}\right)\right)$$

$$\mathbf{E} = -\boldsymbol{\eta}\left(\hat{\mathbf{r}}\times\mathbf{H}\right)$$

Application to practical as well as approximate calculation of fields.

Fields on the surface are approximated and the fields in the volume will be calculated in terms of vector potentials for them. Scattering effects are not exactly but approximately taken into account.

1. Physical optics : Reflected waves are approximated by geometrical optics assuming the frequency is high enough and surface is replaced with infinite plain one. $\boldsymbol{J} = 2\boldsymbol{n} \times \boldsymbol{H}, \quad \boldsymbol{m} = \boldsymbol{0}$
2. Aperture antennas or scattering through apertures: Perturbation of fields due to scattering from finite size of window are neglected and fields on S are assumed to be identical with the incident ones. $\boldsymbol{J} = \boldsymbol{n} \times \boldsymbol{H}^i, \quad \boldsymbol{m} = \boldsymbol{E}^i \times \boldsymbol{n}$

Problems

1. Suppose that a plane wave is incident from –z axis. Derive the equivalent surface electric and magnetic currents on the x-y plane at z = 0 and calculate the fields from these surface currents in the region z > 0 and z < 0. Prove and explain the field equivalence principle.
2. Suppose that a plane wave is incident from –z axis. At z = 0 and in the x-y plane, a rectangular square plate of electrical conductor (2a x 2b) is reflecting it. Derive the approximate surface electric and magnetic currents on plate and calculate the fields from these surface currents in the far field region for z > 0 and z < 0. Predict qualitatively the geometrical optics behavior of the total field. (Reflection, Standing wave, shadow etc.)

Maxwell equations

$$\nabla \times \mathbf{H} = \; j\omega\varepsilon \mathbf{E} + \mathbf{J}$$
$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H} - \mathbf{M}$$

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon}$$

$$\nabla \bullet \mathbf{H} = \frac{\rho^*}{\varepsilon}$$

duality

$$\varepsilon \rightarrow \mu$$
$$\mu \rightarrow \varepsilon$$
$$\mathbf{E} \rightarrow -\mathbf{H}$$
$$\mathbf{H} \rightarrow \mathbf{E}$$
$$\mathbf{J} \rightarrow -\mathbf{M}$$
$$\mathbf{M} \rightarrow \mathbf{J}$$
$$\rho^* \rightarrow \rho$$
$$\rho \rightarrow -\rho^*$$

$\mathbf{A} \triangleq \frac{\mu}{4\pi} \int_v \mathbf{J} \frac{e^{-jkr}}{r} dv$	$\mathbf{B}' \triangleq \frac{\varepsilon}{4\pi} \int_v -\mathbf{M} \frac{e^{-jkr}}{r} dv$
$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$	$\mathbf{E} = \frac{1}{\varepsilon} \nabla \times \mathbf{B}'$
$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla(\nabla \cdot \mathbf{A})}{j\omega\varepsilon\mu}$	$-\mathbf{H} = -j\omega\mathbf{B}' + \frac{\nabla(\nabla \cdot \mathbf{B}')}{j\omega\varepsilon\mu}$

$$\mathbf{E} = \overbrace{-j\omega\mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}}^{-j\omega\mathbf{A}_\perp} - \overbrace{\frac{\nabla \times \mathbf{B}}{\varepsilon}}^{-j\omega\mathbf{B}_\perp \times \hat{\mathbf{r}}\,\eta}$$

⇨ $\mathbf{B} = -\mathbf{B}'$

$$\mathbf{H} = \underbrace{-j\omega\mathbf{B} + \frac{\nabla\nabla \cdot \mathbf{B}}{j\omega\varepsilon\mu}}_{-j\omega\mathbf{B}_\perp} + \underbrace{\frac{\nabla \times \mathbf{A}}{\mu}}_{\hat{\mathbf{r}} \times \frac{-j\omega\mathbf{A}_\perp}{\eta}}$$

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \mu \\ \varepsilon \end{bmatrix} \iiint_v \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix} \frac{e^{-jkr}}{r} dv$$