4. Field equivalence theorem in vector electromagnetic waves.

The previous results show that the surface integral produces the contribution with (-) sign of the direct contribution from the source inside. This will be verified.

(22) $\mathbf{E}(x',y',z') = -\frac{1}{4\pi}\int_s \left[-j\omega\mu(\hat{\mathbf{n}}\times\mathbf{H})\phi + (\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi + (\hat{\mathbf{n}}\cdot\mathbf{E})\nabla\phi\right] da$

(23) $\mathbf{K} = -\hat{\mathbf{n}}\times\mathbf{H},\quad \mathbf{K}^* = \hat{\mathbf{n}}\times\mathbf{E},\quad \eta = -\varepsilon\hat{\mathbf{n}}\cdot\mathbf{E}$ $\quad\hat{\mathbf{n}}$：外向き

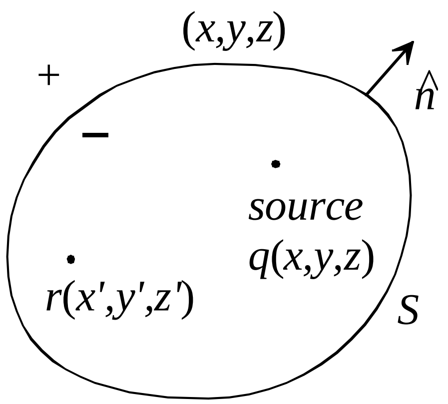
(24) $\mathbf{E}_3(x',y',z') = \frac{1}{4\pi}\int_s \frac{\eta}{\varepsilon}\nabla\phi da$

(C)より $\quad \hat{\mathbf{n}}\cdot\mathbf{E}_3\big|_-^+ = \frac{\eta}{\varepsilon}$

(25) $\mathbf{E}_2(x',y',z') = \frac{1}{4\pi}\int_s \mathbf{K}^*\times\nabla\phi\, da$

(11)より $\quad n\times\mathbf{E}_2\big|_-^+ = \mathbf{K}^*$

(26) $\nabla'\times\mathbf{E}(x',y',z') = +\frac{j\omega\mu}{4\pi}\int_s (\hat{\mathbf{n}}\times\mathbf{H})\times\nabla\phi\, da$



$$\hat{\mathbf{n}}\cdot\mathbf{E}\big|_-^+ = -(\mathbf{n}\cdot\mathbf{E}_-) \qquad \therefore \hat{\mathbf{n}}\cdot\mathbf{E}_+ = 0$$
$$\hat{\mathbf{n}}\times\mathbf{E}\big|_-^+ = -\hat{\mathbf{n}}\times\mathbf{E}_- \qquad \therefore \hat{\mathbf{n}}\times\mathbf{E}_+ = 0$$

These are verified as follows.

$$\mathbf{E} = \frac{1}{4}\oint\left[-j\omega\mu\underbrace{(\mathbf{H}\times\hat{\mathbf{n}})\phi}_{\mathbf{E}_1} - \underbrace{(\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi}_{\mathbf{E}_2} - \underbrace{(\hat{\mathbf{n}}\cdot\mathbf{E})\nabla\phi}_{\mathbf{E}_3}\right] ds$$

$$\begin{cases} \hat{\mathbf{n}}\cdot\mathbf{E}_3\big|_-^+ = -\hat{\mathbf{n}}\cdot\mathbf{E}_- & \text{(C)} \\ \hat{\mathbf{n}}\cdot\mathbf{E}_1\big|_-^+ = 0 & \text{(A)} \\ \hat{\mathbf{n}}\cdot\mathbf{E}_2\big|_-^+ = 0 & \text{(8)} \end{cases}$$

$$\hat{n}\cdot \boldsymbol{E}\big|_-^+ = \hat{n}\cdot(\boldsymbol{E}_1+\boldsymbol{E}_2+\boldsymbol{E}_3)\big|_-^+ = -\hat{n}\cdot\boldsymbol{E}_- + 0 \qquad \therefore \underline{\hat{\mathbf{n}}\cdot\mathbf{E}_+ = 0}$$

$$\left.\begin{cases} \hat{\mathbf{n}}\times\mathbf{E}_2\big|_-^+ = -\hat{\mathbf{n}}\times\mathbf{E}_- & \text{(11)} \\ \hat{\mathbf{n}}\times\mathbf{E}_3\big|_-^+ = 0 & \text{(C)} \\ \hat{\mathbf{n}}\times\mathbf{E}_1\big|_-^+ = 0 & \text{(A)} \end{cases}\right\} \quad \mathbf{E}_+ = 0$$

$$\hat{\mathbf{n}}\times\mathbf{E}\big|_-^+ = -\mathbf{n}\times\mathbf{E}_- \qquad \therefore \underline{\hat{\mathbf{n}}\times\mathbf{E}_+ = 0}$$

Similarly,

同様に $\left.\begin{array}{l}\hat{\mathbf{n}}\cdot\mathbf{H}_{+}=0\\ \hat{\mathbf{n}}\times\mathbf{H}_{+}=0\end{array}\right\}$

∴ 外部の観測点に対して $\int_v + \int_s \equiv 0 \xrightarrow{S外} \int_v$ を打ち消す働き

For observer outside, canceling effects. $\mathbf{E}\times\hat{\mathbf{n}}=0\,,\,\mathbf{E}\cdot\hat{\mathbf{n}}=0$

$\mathbf{H}\times\hat{\mathbf{n}}=0\,,\,\mathbf{H}\cdot\hat{\mathbf{n}}=0$

Q.E.D.

Field equivalence theorem for EM fields

$$\mathbf{E}=\frac{1}{4\pi}\int\left(\underline{-j\omega\mu\phi\,\mathbf{J}}-\underline{\mathbf{J}^{*}\times\nabla\phi}+\underline{\nabla\phi\frac{\rho}{\varepsilon}}\right)dv$$

*1, *2, *3 (Three terms)

$$+\frac{1}{4\pi}\oint\left\{\underline{-j\omega\mu\phi\left(\mathbf{H}\times\hat{\mathbf{n}}\right)}-\underline{\left(\hat{\mathbf{n}}\times\mathbf{E}\right)\times\nabla\phi}-\underline{\nabla\phi\left(\hat{\mathbf{n}}\cdot\mathbf{E}\right)}\right\}ds$$

*4, *5, *6(Three terms)

$$\mathbf{H}=\frac{1}{4\pi}\int_v\left(\underline{-j\omega\varepsilon\,\mathbf{J}^{*}\phi}+\underline{\mathbf{J}\times\nabla\phi}+\underline{\nabla\phi\frac{\rho^{*}}{\mu}}\right)dv$$

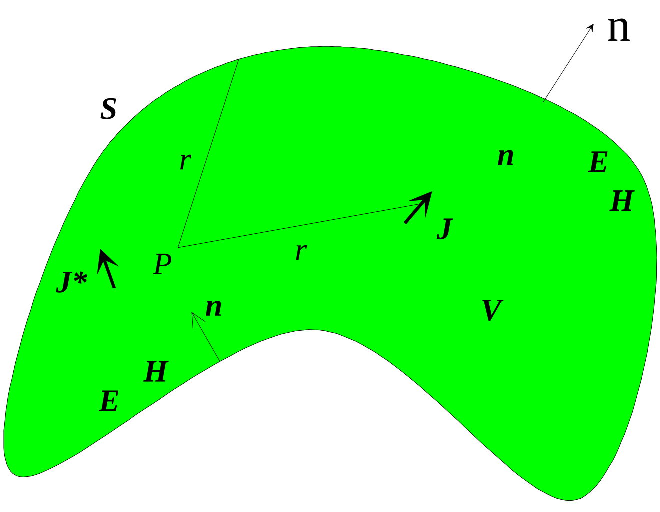
*1, *2, *3 (Three terms)

$$+\frac{1}{4\pi}\oint_s\left\{\underline{-j\omega\varepsilon\phi\left(\hat{\mathbf{n}}\times\mathbf{E}\right)}-\underline{\left(\hat{\mathbf{n}}\times\mathbf{H}\right)\times\nabla\phi}-\underline{\nabla\phi\left(\hat{\mathbf{n}}\cdot\mathbf{H}\right)}\right\}ds$$

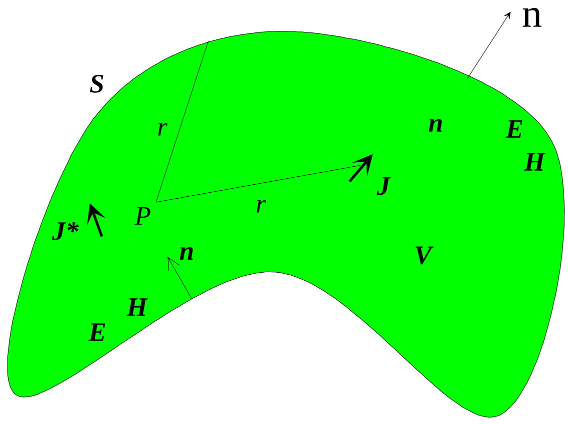
*4, *5, *6(Three terms)

If there are no source outside of *V*, *4 + *5 + *6 = 0 for observer inside of *V*.
If there are no source inside of *V*, *4 + *5 + *6 = 0 for observer outside of *V*.

For observer outside of *V*, *1+ *2+ *3+ *4 + *5 + *6 = 0,
irrespective of environment outside of *V*.



Introduction of equivalent surface currents defined by tangential electric and magnetic fields.



After substituting

$$J^* = n \times E$$
$$J = H \times n$$
$$\rho^* = -\mu(n \cdot H)$$
$$\rho = -\varepsilon(n \cdot E)$$

We have followings.

$$E = \frac{1}{4\pi}\int_{s+v}\left(-j\omega\mu\phi J - J^* \times \nabla\phi + \nabla\phi\frac{\rho}{\varepsilon}\right)dv$$
$$H = \frac{1}{4\pi}\int_{s+v}\left(-j\omega\varepsilon\phi J^* + J \times \nabla\phi + \nabla\phi\frac{\rho^*}{\mu}\right)dv$$

第３項を*Tangential*で表す。 → （A3）

$$\frac{1}{4\pi}\oint\left(-\nabla\phi\left(n\cdot E\right)_{\_}\right)ds = \frac{1}{4\pi}\oint\left(-\frac{1}{j\omega\varepsilon}\left[\left(n\times H\right)\cdot\nabla\right]\nabla\phi\right)ds$$
$$\frac{1}{4\pi}\oint\left(-\nabla\phi\left(n\cdot H\right)_{\_}\right)ds = \frac{1}{4\pi}\oint\left(+\frac{1}{j\omega\mu}\left[\left(n\times E\right)\cdot\nabla\right]\nabla\varphi\right)ds$$

$$E = \frac{1}{4\pi}\int_v\left(-j\omega\mu\phi J - J^* \times \nabla\phi + \nabla\phi\frac{\rho}{\varepsilon}\right)dv$$
$$+\frac{1}{4\pi}\int_s\left\{-j\omega\mu\phi\left(H_{\_}\times n\right) - \left(n\times E\right)\times\nabla\phi + \frac{1}{j\omega\varepsilon}\left[\left(H_{\_}\times n\right)\cdot\nabla\right]\nabla\phi\right\}ds$$
$$H = \frac{1}{4\pi}\int_v\left(-j\omega\varepsilon\phi J^* + J \times \nabla\phi + \nabla\phi\frac{\rho^*}{\mu}\right)dv$$
$$+\frac{1}{4\pi}\int_s\left\{-j\omega\varepsilon\phi\left(n\times E_{\_}\right) - \left(n\times H\right)\times\nabla\phi + \frac{1}{j\omega\mu}\left[\left(n\times E_{\_}\right)\cdot\nabla\right]\nabla\phi\right\}ds$$

We can conclude that

Fields can be uniquely determined by the Sources inside V and on S only.

$$\left.\begin{array}{ll} v\text{内の} & J, J^* \\ s\text{上の} & N\times E, n\times H \end{array}\right\}$$ のみで決定される。

ds, dv, V は全て積分座標系。

$J_s \equiv H_- \times n$　　$\rho_s^* = -\mu n \cdot H$

n は外向き

$J_s^* \equiv n \times E_-$　　$\rho_s = -\varepsilon n \cdot E$

s 内に波源がない時 if there are no source in V

$$E = \frac{1}{4\pi}\int_s \left\{ -j\omega\mu\phi J_s - J_s^* \times \nabla\phi + \nabla\phi \frac{\rho_s}{\varepsilon} \right\} dv$$

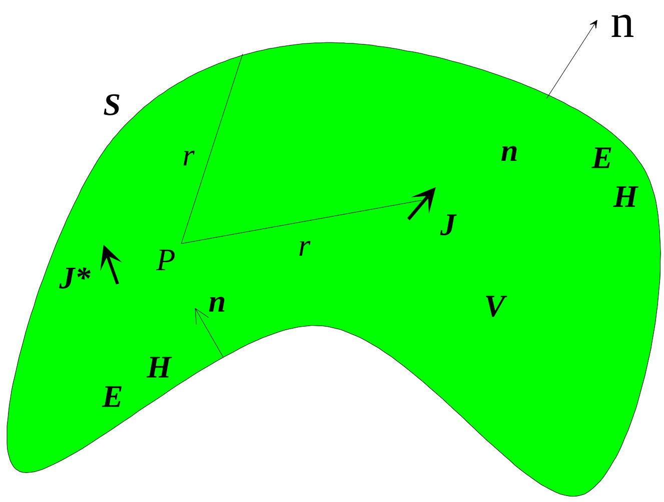
$$H = \frac{1}{4\pi}\int_s \left\{ -j\omega\varepsilon\phi J_s^* + J_s \times \nabla\phi + \nabla\phi \frac{\rho_s^*}{\mu} \right\} dv$$

$$J_s = n_i \times H \qquad \rho_s^* = n_i \cdot H\mu \qquad A_s = \frac{\mu}{4\pi}\oint J_s \phi ds$$

$$J_s^* = E \times n_i \qquad \rho_s = \varepsilon\, n_i \cdot E \qquad B_s = \frac{\varepsilon}{4\pi}\oint J_s^* \phi ds$$

$$E = -j\omega A_s + \frac{\nabla\nabla\cdot A_s}{j\omega\varepsilon\mu} - \frac{\nabla\times B_s}{\varepsilon}, \quad H = \frac{\nabla\times A_s}{\mu} + \frac{\nabla\nabla\cdot B_s}{j\omega\varepsilon\mu} - j\omega B_s$$

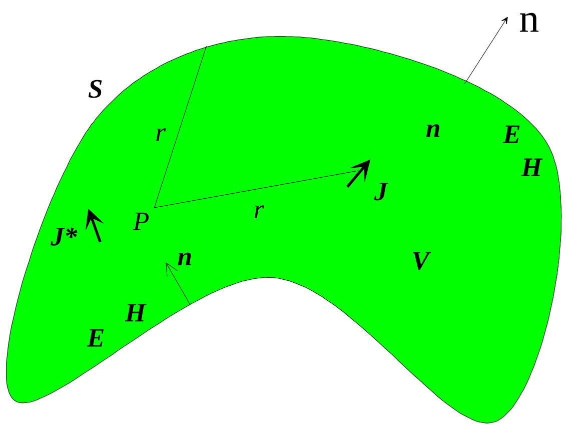
Ref. Duality



References 参考:

$\nabla \times H = j\omega\varepsilon E + J$

$\nabla \times E = -j\omega\mu H$



Vector potential

$\mu H \equiv \nabla \times A$

Vector Helmholtz Equation

$\nabla^2 A + k^2 A = -\mu J$

Solution

$$A = \frac{\mu}{4\pi}\int_V \mathbf{J}\frac{e^{-jkr}}{r}dv + \frac{\mu}{4\pi}\iint_S \frac{\partial}{\partial n}\mathbf{A}\cdot\frac{e^{-jkr}}{r}ds - \frac{\mu}{4\pi}\iint_S \mathbf{A}\cdot\frac{\partial}{\partial n}\frac{e^{-jkr}}{r}ds$$

If 放射条件 $r\left\{\frac{\partial}{\partial r}\mathbf{A} + jk\mathbf{A}\right\}_{r=\infty} \to 0$ （外に波源、散乱体がない）

Free space Green's Function

$$A = \frac{\mu}{4\pi}\int_v J\frac{e^{-jkr}}{r}dv$$

Fields are expressed in terms of vector potential **A** as:

$$H = \frac{1}{\mu}\nabla \times A$$

$$E = -j\omega A + \underline{\frac{\nabla(\nabla\cdot A)}{j\omega\varepsilon\mu}}$$

$$\downarrow$$

$$-\nabla\phi$$

When only the far fields are considered, we have a TEM wave.

$|r| \to \infty$ の中は簡単になり TEM波

$$E = -j\omega\left(A - \hat{r}\left(A\cdot\hat{r}\right)\right)$$

$$H = \frac{1}{\eta}\left(\hat{r}\times E\right) \qquad \eta = \sqrt{\mu/\varepsilon} \approx 120\pi$$

Appendix n：外向き

（A1）

$$\int \nabla\cdot\left(P\times\nabla\times Q\right)dv=\int\left(P\times\nabla\times Q\right)\cdot nds$$

$\nabla\cdot\left(A\times B\right)=B\cdot\nabla\times A-A\cdot\nabla\times B$ を用いて

$$\int \nabla\times Q\cdot\nabla\times Pdv-\int P\cdot\nabla\times\nabla\times Qdv=\int\left(P\times\nabla\times Q\right)\cdot ds \qquad (\text{I})$$

$P\Rightarrow Q$

$$\int \nabla\times P\cdot\nabla\times Qdv-\int Q\cdot\nabla\times\nabla\times Pdv=\int\left(Q\times\nabla\times P\right)\cdot ds \qquad (\text{II})$$

（Ⅰ）－（Ⅱ）

$$\int Q\cdot\nabla\times\nabla\times Pdv-\int P\cdot\nabla\times\nabla\times Qdv=\int\left[\left(P\times\nabla\times Q\right)-\left(Q\times\nabla\times P\right)\right]ds$$

Q.E.D

（A2）任意ベクトル arbitrary vector：a

$$div\left(\phi J^*\times a\right)=a\cdot\nabla\times\left(\phi J^*\right)-\phi J^*\cdot\underset{0}{\underline{\nabla\times a}}$$

$$=a\cdot\left\{\phi\nabla\times J^*-J^*\times\nabla\phi\right\}$$

$$\therefore \int div\left(\phi J^*\times a\right)dv=\int\left(\phi J^*\times a\right)ds'=\int a\cdot\left\{\phi\nabla\times J^*-J^*\times\nabla\phi\right\}dv$$

$$\int \phi J^*\times a\cdot nds=-\int a\cdot\left(\phi J^*\times n\right)ds=\int a\cdot\left\{\phi\nabla\times J^*-J^*\times\nabla\phi\right\}dv$$

∴ aを除くと

$$\int n\times\left(\phi J^*\right)ds=\int\left\{\phi\nabla\times J^*-J^*\times\nabla\phi\right\}dv$$

Q. E. D

（A3）

参考 $E=-j\omega A+\dfrac{\nabla\nabla\cdot A}{j\omega\varepsilon\mu}=-j\varpi\left(A+\dfrac{\nabla\nabla\cdot A}{k^2}\right) \qquad \left(\phi=\dfrac{e^{-jkr}}{r}\right)$

$$A=\frac{\mu}{4\pi}\int\left(H\times n\right)\cdot\frac{e^{-jkr}}{r}dv'=\frac{\mu}{4\pi}\int\left(H\times n\right)\phi\, dv'$$

ところで *By the way*

$$E=\int v+\frac{1}{4\pi}\oint\left\{-j\omega\mu\phi\left(H\times n\right)-\nabla\phi\left(n\cdot E\right)\right\}ds$$

$$=\int v+\left(-j\omega\right)\oint\left\{\frac{1}{4\pi}\left(H\times n\right)\phi+\frac{\nabla\phi\left(n\cdot E\right)}{j\omega 4\pi}\right\}ds \quad \text{より}$$

$$\frac{\nabla\nabla\cdot A}{k^2}=\frac{1}{j\omega 4\pi}\oint\nabla\phi\left(n\cdot E\right)ds$$ を証明する。 *will be verified*.

$$\nabla A=\frac{\mu}{4\pi}\int\nabla\left(H\times n\right)\phi ds=\frac{\mu}{4\pi}\int\left(H\times n\right)\cdot\nabla\phi ds \qquad \because\ \nabla A\phi=A\cdot\nabla\phi+\phi\cdot\underset{0}{\underline{\nabla A}}$$

$$\because\ \nabla\left(\nabla A\right)\phi=\nabla\left(A\cdot\nabla\phi\right)=A\times\left(\underset{0}{\underline{\nabla\times\nabla\phi}}\right)+\left(A\cdot\nabla\right)\nabla\phi=\left(A\cdot\nabla'\right)\nabla'\phi \quad \text{より}$$

$$=-\frac{\mu}{4\pi}\int n\cdot\left(H\times\infty\phi\right)ds$$

$$=\frac{\mu}{4\pi}\int\phi\underset{j\omega\varepsilon E}{\underline{\nabla\times H}}\cdot nds$$

$$=\frac{j\omega\varepsilon\mu}{4\pi}\int\phi E\cdot nds$$

ところで By the way ,

$$\nabla\times\phi H=\phi\nabla\times H+\nabla\phi\times H$$

$$\therefore\ \int\nabla\times\left(\phi H\right)\cdot nds=\int div\left(\nabla\times\phi H\right)dv=0$$

$$\therefore\ \int\phi\nabla H\cdot nds=-\int\nabla\phi\times H\cdot nds$$

$$\therefore\ \frac{\nabla\nabla\cdot A}{k^2}=\frac{j\mu\alpha\mu}{4\pi}\int\nabla\phi(n\cdot E)\ ds=\frac{j}{4\pi\omega}\int\nabla\phi(n\cdot E)\ ds$$

$$=\frac{j}{4\pi\omega}\int\nabla'\phi(n\cdot E)\ ds$$

↓

積分系

$$\therefore\ \frac{1}{4\pi}\oint\left\{-\nabla'\phi(n\cdot E)\right\}ds=\frac{\nabla\nabla\cdot A}{k^2}(-j\omega)=\frac{-j}{4\pi\omega\varepsilon}\oint\left\{(H\times n)\nabla\right\}\nabla\phi\ ds$$

$$=\frac{-1}{4\pi j\omega\varepsilon}\oint\left\{(n\times H_{=})\nabla'\right\}\nabla'\phi\ ds$$