3. Direct Integration of the Field Equations

◆ *Vector equivalent theorem*　　∇は積分座標系

公式　$\int_v \left(Q\cdot\nabla\times\nabla\times P - P\cdot\nabla\times\nabla\times Q\right)dv = \oint_s (P\times\nabla\times Q - Q\times\nabla\times P)\cdot n\, da$　（A1）

Vector Algebra　　***ベクトル解析の公式***　　ｎ：外向き

V,U　*scalar* スカラ量　**A,B**　*vector* ベクトル量

A）ベクトルの微分公式

$$\nabla(VU) = V\nabla U + U\nabla V$$
$$\nabla(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B}$$
$$\nabla\cdot\nabla V = \nabla^2 V$$
$$\nabla\cdot(V\mathbf{A}) = V\nabla\cdot\mathbf{A} + \mathbf{A}\nabla\cdot V$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$$
$$\nabla\cdot\nabla\times\mathbf{A} = 0$$
$$\nabla\times(\nabla V) = 0$$
$$\nabla\times(V\mathbf{A}) = V\nabla\times\mathbf{A} + \nabla V\times\mathbf{A}$$
$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

◆ *Maxwell's Equations*

Duality of ***E*** and ***H***

$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}^*$	$\boldsymbol{E}\to\boldsymbol{H}$　　$\varepsilon\leftrightarrow\mu$
$\nabla\times\mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J}$	$\boldsymbol{H}\to -\boldsymbol{E}$
$\nabla\cdot\mathbf{H} = \dfrac{\rho^*}{\mu}$	$\boldsymbol{J}^*\to -\boldsymbol{J}$
	$\boldsymbol{J}\to\boldsymbol{J}^*$
$\nabla\cdot\mathbf{E} = \dfrac{\rho}{\varepsilon}$	$\rho\to\rho^*$　　$\rho^*\to -\rho$
$\nabla\cdot\mathbf{J} + j\omega\rho = 0$	
$\nabla\cdot\mathbf{J}^* + j\omega\rho^* = 0$	

$$\nabla\times\nabla\times\mathbf{E} = -j\omega\mu\left(j\omega\varepsilon\mathbf{E} + \mathbf{J}\right) - \nabla\times\mathbf{J}^* = k^2\mathbf{E} - \nabla\times\mathbf{J}^* - j\omega\mu\mathbf{J}$$
$$\nabla\times\nabla\times\mathbf{H} = j\omega\varepsilon\left(-j\omega\mu\mathbf{H} - \mathbf{J}^*\right) + \nabla\times\mathbf{J} = k^2\mathbf{H} + \nabla\times\mathbf{J} - j\omega\varepsilon\mathbf{J}^*$$

Maxwell equation and Vector equivalent theorem → Direct Integration

$\mathbf{P} \equiv \mathbf{E}$, $\mathbf{Q} \equiv \phi\,\mathbf{a}$ $\quad \nabla^2\phi + k^2\phi = 0$ $\quad a$：定ベクトル

$\nabla^2\mathbf{Q} + k^2\mathbf{Q} = 0$

Maxwell equation

$$\nabla\times\nabla\times\mathbf{Q} = \mathbf{a}\,k^2\phi + \nabla\left(\mathbf{a}\cdot\nabla\phi\right) \qquad \because\ \nabla\times\left(\phi\ \mathbf{a}\right) = \phi\,\nabla\times\mathbf{a} + \nabla\phi\times\mathbf{a} = \nabla\phi\times\mathbf{a}$$

$$\nabla\times\nabla\times\mathbf{P} = k^2\mathbf{E} - \nabla\times\mathbf{J}^* - j\omega\mu\mathbf{J} \qquad \therefore\ \nabla\times\left(\nabla\times\mathbf{Q}\right) = \nabla\phi\nabla\cdot\mathbf{a} - \mathbf{a}\nabla\cdot\nabla\phi + \left(\mathbf{a}\cdot\nabla\right)\nabla\phi - \left(\nabla\phi\cdot\right)\mathbf{a}$$

Vector equivalent theorem

左辺: $\phi\,\mathbf{a}\cdot\left(k^2\mathbf{E} - \nabla\times\mathbf{J}^* - j\omega\mu\,\mathbf{J}\right) - \mathbf{E}\cdot\left(\mathbf{a}\,k^2\phi + \nabla\left(\mathbf{a}\cdot\nabla\phi\right)\right)$

$$= \int_v -\phi\ \ \mathbf{a}\cdot\nabla\times\mathbf{J}^* - j\omega\mu\phi\,\mathbf{a}\cdot\mathbf{J} - \mathbf{E}\cdot\nabla\left(\mathbf{a}\cdot\nabla\phi\right)dv$$

$div\left\{\mathbf{E}\left(\mathbf{a}\cdot\nabla\phi\right)\right\} = \mathbf{E}\cdot\nabla\left(\mathbf{a}\cdot\nabla\phi\right) + \left(\mathbf{a}\cdot\nabla\phi\right)\nabla\cdot\mathbf{E}$ より

$$\int div\mathbf{E}\left(\mathbf{a}\cdot\nabla\phi\right)dv = \int\mathbf{E}\left(\mathbf{a}\cdot\nabla\phi\right)\cdot ds = \int\mathbf{E}\cdot\nabla\left(\mathbf{a}\cdot\nabla\phi\right)dv + \int\left(\mathbf{a}\cdot\nabla\phi\right)\nabla\cdot\mathbf{E}dv$$

$$\therefore\ \int\mathbf{E}\cdot\nabla\left(\mathbf{a}\cdot\nabla\phi\right)dv = \int\mathbf{E}\left(\mathbf{a}\cdot\nabla\phi\right)ds - \int\left(\mathbf{a}\cdot\nabla\phi\right)\frac{\rho}{\varepsilon}dv$$

$$左辺 = \int_v\left(-\phi\,\mathbf{a}\cdot\nabla\times\mathbf{J}^* - j\omega\mu\phi\,\mathbf{a}\cdot\mathbf{J} + \left(\mathbf{a}\cdot\nabla\phi\right)\frac{\rho}{\varepsilon}\right)dv - \oint_s\mathbf{E}\left(\mathbf{a}\cdot\nabla\phi\right)ds$$

右辺: $\left(\mathbf{P}\times\nabla\times\mathbf{Q} - \mathbf{Q}\times\nabla\times\mathbf{P}\right)\cdot\hat{\mathbf{n}}$

$$\mathbf{P}\times\nabla\times\mathbf{Q} = \mathbf{E}\times\left(\nabla\times\phi\,\mathbf{a}\right) = \mathbf{E}\times\left(\phi\nabla\times\mathbf{a} + \nabla\phi\times\mathbf{a}\right) = \mathbf{E}\times\left(\nabla\phi\times\mathbf{a}\right) = \nabla\phi\cdot\left(\mathbf{E}\cdot\mathbf{a}\right) - \mathbf{a}\cdot\left(\nabla\phi\cdot\mathbf{E}\right)$$

$$\mathbf{Q}\times\nabla\times\mathbf{P} = \phi\,\mathbf{a}\times\left(-j\omega\mu\mathbf{H} - \mathbf{J}^*\right)$$

$$右辺 = \left(\hat{\mathbf{n}}\cdot\nabla\phi\right)\left(\mathbf{E}\cdot\mathbf{a}\right) - \left(\hat{\mathbf{n}}\cdot\mathbf{a}\right)\left(\nabla\phi\cdot\mathbf{E}\right) + j\omega\mu\phi\,\mathbf{a}\times\mathbf{H}\cdot\hat{\mathbf{n}} + \phi\,\mathbf{a}\times\mathbf{J}^*\cdot\hat{\mathbf{n}}$$

$$= \underline{\mathbf{a}\cdot\mathbf{E}\left(\hat{\mathbf{n}}\nabla\phi\right) - \mathbf{a}\cdot\hat{\mathbf{n}}\left(\mathbf{E}\cdot\nabla\phi\right)} + j\omega\mu\phi\,\mathbf{a}\cdot\left(\mathbf{H}\times\hat{\mathbf{n}}\right) + \phi\,\mathbf{a}\cdot\mathbf{J}^*\times\hat{\mathbf{n}}$$

$$\mathbf{a}\cdot\left(\hat{\mathbf{n}}\times\mathbf{E}\right)\times\nabla\phi$$

$$右辺 = \oint_s\mathbf{a}\cdot\left(\hat{\mathbf{n}}\times\mathbf{E}\right)\times\nabla\phi + j\omega\mu\phi\,\mathbf{a}\cdot\left(\mathbf{H}\times\hat{\mathbf{n}}\right) + \phi\,\mathbf{J}^*\times\hat{\mathbf{n}}ds$$

右辺＝左辺

$$\therefore \int \mathbf{a}\cdot\left\{-\phi\,\nabla\times\mathbf{J}^{*}-j\omega\mu\phi\,\mathbf{J}+\nabla\phi\frac{\rho}{\varepsilon}\right\}dv$$
$$=\oint \mathbf{a}\cdot\left\{(\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi+j\omega\mu\phi(\mathbf{H}\times\hat{\mathbf{n}})+\phi\,\mathbf{J}^{*}\times\hat{\mathbf{n}}+\nabla\phi(\hat{\mathbf{n}}\cdot\mathbf{E})\right\}ds$$

a：任意 *arbitrary*

$$\int\left(-\phi\,\nabla\times\mathbf{J}^{*}-j\omega\mu\phi\,\mathbf{J}+\nabla\phi\frac{\rho}{\varepsilon}\right)dv$$
$$=\oint\left\{j\omega\mu\phi(\mathbf{H}\times\hat{\mathbf{n}})+(\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi+\phi\,\mathbf{J}^{*}\times\hat{\mathbf{n}}+\nabla\phi(\hat{\mathbf{n}}\cdot\mathbf{E})\right\}ds$$

On the otherhand ところで

$$\oint\phi\,\mathbf{J}^{*}\times\hat{\mathbf{n}}ds=-\int_{v}\left(\left(\phi\nabla\times\mathbf{J}^{*}\right)-\mathbf{J}^{*}\times\nabla\phi\right)dv \qquad \text{(A2)}$$

$$\therefore \int -\mathbf{J}^{*}\times\nabla\phi-j\omega\mu\phi\mathbf{J}+\nabla\phi\frac{\rho}{\varepsilon}dv$$
$$=\oint\left\{j\omega\mu\phi(\mathbf{H}\times\hat{\mathbf{n}})+(\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi+\nabla\phi(\hat{\mathbf{n}}\cdot\mathbf{E})\right\}ds$$

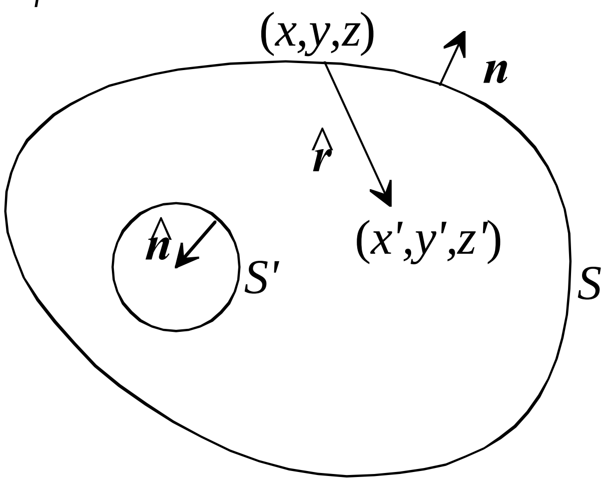
Extraction Singularities 特異点の抽出

・*Definition gradient*

(x,y,z)がρ方向へ動く　rが小さく、$1/r$が大きく
∴　ベクトルはρ方向を向いている

$$on\ S' \quad \phi = \frac{e^{-jkr}}{r} \qquad (x,y,z)系 \rightarrow \quad \nabla\phi = \nabla\left(\frac{e^{-jkr}}{r}\right) = \frac{\hat{\mathbf{n}}}{r^2}$$

$$\nabla\phi = \frac{\partial}{\partial r}\phi(-\hat{\mathbf{n}}) = \frac{-jkr-1}{r^2}e^{-jkr}(-\hat{\mathbf{n}})$$
$$= \left(\frac{1}{r^2} + jk\frac{1}{r}\right)e^{-jkr}\hat{\mathbf{n}}$$
$$\rightarrow \frac{\hat{\mathbf{n}}}{r^2}$$



$$\int ds \cong 4\pi r^2$$

$$\oint_{S'} \rightarrow \oint_{S'}\left\{j\omega\mu\frac{1}{r}(\mathbf{H}\times\hat{\mathbf{n}}) + \frac{1}{r^2}\left[(\hat{\mathbf{n}}\times\mathbf{E})\times\hat{\mathbf{n}} + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{E})\right]\right\}ds \quad (r\rightarrow 0)$$
$$\rightarrow 4\pi\mathbf{E}(x_0,y_0,z_0)$$

$$\therefore \quad \mathbf{E} = \frac{1}{4\pi}\int_V\left(-j\omega\mu\phi\,\mathbf{J} - \mathbf{J}^*\times\nabla\phi + \nabla\phi\frac{\rho}{\varepsilon}\right)dv$$
$$+\frac{1}{4\pi}\oint_S\left\{-j\omega\mu\phi(\mathbf{H}\times\hat{\mathbf{n}}) - (\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi - \nabla\phi(\hat{\mathbf{n}}\cdot\mathbf{E})\right\}ds$$

note:

$$\nabla\phi\frac{\rho}{\varepsilon} = \frac{\nabla\phi\nabla\bullet\mathbf{J}}{-j\omega\varepsilon} \qquad \because \nabla\bullet\mathbf{J} = -j\omega\rho$$

By the way,

$$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkr}}{r}dV$$

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla(\nabla\cdot\mathbf{A})}{j\omega\varepsilon\mu}$$

第１項と第２項と比べて　*Comparing the first and second terms,*

$$\mathbf{J} = \mathbf{H}\times\hat{\mathbf{n}}$$
$$\mathbf{J}^* = \hat{\mathbf{n}}\times\mathbf{E}$$
$$\rho = -\varepsilon(\hat{\mathbf{n}}\cdot\mathbf{E}) \qquad \hat{\mathbf{n}}:外向き \ \ outer\ normal$$

From the duality of $\boldsymbol{E}$ and $\boldsymbol{H}$,

$$\nabla\times\mathbf{E}=-j\omega\mu\mathbf{H}-\mathbf{J}^{*} \qquad \mu\to\varepsilon \qquad \nabla\times(-\mathbf{H})=-j\omega\varepsilon\mathbf{E}-\mathbf{J}$$

$$\nabla\times\mathbf{H}=j\omega\varepsilon\mathbf{E}+\mathbf{J} \qquad \varepsilon\to\mu \qquad \nabla\times(\mathbf{E})=j\omega\mu(-\mathbf{H})-\mathbf{J}^{*}$$

$$\nabla\cdot\mathbf{H}=\frac{\rho^{*}}{\mu} \qquad \mathbf{E}\to-\mathbf{H}$$

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon} \qquad \mathbf{H}\to\mathbf{E}$$

$$\mathbf{J}^{*}\to\mathbf{J}$$

$$\mathbf{J}\to-\mathbf{J}^{*}$$

$$\rho\to-\rho^{*}$$

$$\rho^{*}\to\rho$$

On the other hand, from the duality

$$\mathbf{H}=\frac{1}{4\pi}\int_{v}\left(-j\omega\varepsilon\,\mathbf{J}^{*}\phi+\mathbf{J}\times\nabla\phi+\nabla\phi\frac{\rho^{*}}{\mu}\right)dv$$

$$+\frac{1}{4\pi}\oint_{s}\left\{-j\omega\varepsilon\phi(\hat{\mathbf{n}}\times\mathbf{E})-(\hat{\mathbf{n}}\times\mathbf{H})\times\nabla\phi-\nabla\phi(\hat{\mathbf{n}}\cdot\mathbf{H})\right\}ds$$

$$\boldsymbol{J}^{*}=\hat{\boldsymbol{n}}\times\boldsymbol{E}$$

$$\boldsymbol{J}=\boldsymbol{H}\times\hat{\boldsymbol{n}}$$

$$\rho^{*}=-\mu(\hat{\boldsymbol{n}}\cdot\boldsymbol{H})$$

Integral Expression of EM Fields

$$\mathbf{E}=\frac{1}{4\pi}\int\left(-j\omega\mu\phi\,\mathbf{J}-\mathbf{J}^{*}\times\nabla\phi+\nabla\phi\,\frac{\rho}{\varepsilon}\right)dv$$

$$+\frac{1}{4\pi}\oint\left\{-j\omega\mu\phi(\mathbf{H}\times\hat{\mathbf{n}})-(\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi-\nabla\phi(\hat{\mathbf{n}}\cdot\mathbf{E})\right\}ds$$

$$\mathbf{H}=\frac{1}{4\pi}\int_{v}\left(-j\omega\varepsilon\,\mathbf{J}^{*}\phi+\mathbf{J}\times\nabla\phi+\nabla\phi\frac{\rho^{*}}{\mu}\right)dv$$

$$+\frac{1}{4\pi}\oint_{s}\left\{-j\omega\varepsilon\phi(\hat{\mathbf{n}}\times\mathbf{E})-(\hat{\mathbf{n}}\times\mathbf{H})\times\nabla\phi-\nabla\phi(\hat{\mathbf{n}}\cdot\mathbf{H})\right\}ds$$

Compare it with 1-13.