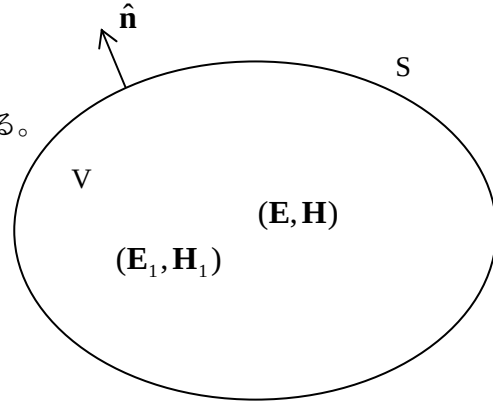


Uniqueness 定理

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} = j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E}$$

V 内に Maxwell の方程式を満たす 2 組の電磁界分布 $(\mathbf{E}, \mathbf{H}), (\mathbf{E}_1, \mathbf{H}_1)$ が存在すると仮定する。
 $\sigma \neq 0$ を仮定する。



ベクトル公式: $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$

において、

$$\mathbf{a} = \mathbf{E} - \mathbf{E}_1$$

$$\mathbf{b} = (\mathbf{H} - \mathbf{H}_1)^*$$

* 複素共役(conjugate)

とおく。

$$\begin{aligned} \nabla \cdot [(\mathbf{E} - \mathbf{E}_1) \times (\mathbf{H} - \mathbf{H}_1)^*] &= (\mathbf{H} - \mathbf{H}_1)^* \cdot \nabla \times (\mathbf{E} - \mathbf{E}_1) - (\mathbf{E} - \mathbf{E}_1) \cdot \nabla \times (\mathbf{H} - \mathbf{H}_1)^* \\ &= (\mathbf{H} - \mathbf{H}_1)^* \cdot (\mathbf{H} - \mathbf{H}_1)(-j\omega\mu) - (\mathbf{E} - \mathbf{E}_1) \cdot (\mathbf{E} - \mathbf{E}_1)^*(j\omega\varepsilon) \\ &\quad - (\mathbf{E} - \mathbf{E}_1) \cdot (\mathbf{E} - \mathbf{E}_1)^* \sigma \end{aligned}$$

$$\begin{aligned} &= -j\omega \left\{ |\mathbf{H} - \mathbf{H}_1|^2 \mu - |\mathbf{E} - \mathbf{E}_1|^2 \varepsilon \right\} - |\mathbf{E} - \mathbf{E}_1|^2 \sigma \\ \iiint_V \nabla \cdot [(\mathbf{E} - \mathbf{E}_1) \times (\mathbf{H} - \mathbf{H}_1)^*] dV &= \oiint_S (\mathbf{E} - \mathbf{E}_1) \times (\mathbf{H} - \mathbf{H}_1)^* \cdot d\mathbf{S} \\ &= -j\omega \iiint_V \left\{ |\mathbf{H} - \mathbf{H}_1|^2 \mu - |\mathbf{E} - \mathbf{E}_1|^2 \varepsilon \right\} dV - \sigma \iiint_V |\mathbf{E} - \mathbf{E}_1|^2 dV \end{aligned}$$

$$\oiint_S (\mathbf{E} - \mathbf{E}_1) \times (\mathbf{H} - \mathbf{H}_1)^* \cdot \hat{\mathbf{n}} dS = -j\omega \iiint_V \left\{ |\mathbf{H} - \mathbf{H}_1|^2 \mu - |\mathbf{E} - \mathbf{E}_1|^2 \varepsilon \right\} dV - \sigma \iiint_V |\mathbf{E} - \mathbf{E}_1|^2 dV$$

$\sigma \neq 0$ とすると、S 上で $\mathbf{E} \times \hat{\mathbf{n}} = \mathbf{E}_1 \times \hat{\mathbf{n}}$ or $\mathbf{H} \times \hat{\mathbf{n}} = \mathbf{H}_1 \times \hat{\mathbf{n}}$ ならば、
 V 内の至る所で $\mathbf{E} = \mathbf{E}_1$ (Real=0) and $\mathbf{H} = \mathbf{H}_1$ (Imag=0)

なぜならば、

$$\{(\mathbf{E} - \mathbf{E}_1) \times (\mathbf{H} - \mathbf{H}_1)\} \cdot \hat{\mathbf{n}} = \{(\mathbf{H} - \mathbf{H}_1) \times \hat{\mathbf{n}}\} \cdot (\mathbf{E} - \mathbf{E}_1) = \{\hat{\mathbf{n}} \times (\mathbf{E} - \mathbf{E}_1)\} \cdot (\mathbf{H} - \mathbf{H}_1)$$

実は、 $\sigma = 0$ の場合も含めて Uniqueness 定理は成立する。

Poynting Vector

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J}$$

公式: $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$

$$\mathbf{a} \Rightarrow \mathbf{E}, \quad \mathbf{b} \Rightarrow \mathbf{H}^*$$

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) &= \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* \\ &= \mathbf{H}^* \cdot (-j\omega\mu\mathbf{H}) - \mathbf{E} \cdot (j\omega\epsilon\mathbf{E} + \mathbf{J}) \\ &= -j\omega\mu\mathbf{H} \cdot \mathbf{H}^* + j\omega\epsilon\mathbf{E} \cdot \mathbf{E}^* - \mathbf{E} \cdot \mathbf{J}^* \end{aligned}$$

$$\mathbf{S} \triangleq \frac{\mathbf{E} \times \mathbf{H}^*}{2}$$

$$\oint_S \mathbf{P} \cdot \hat{\mathbf{n}}' dS = \oint_S \frac{\mathbf{E} \times \mathbf{H}^*}{2} \cdot \hat{\mathbf{n}}' dS = -\iiint_V \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{H}^*}{2} \right) dV$$

内向き

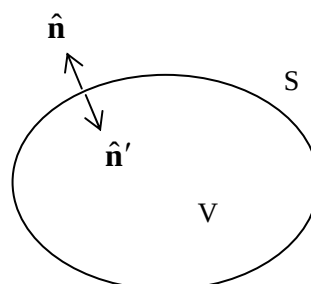
$$= \iiint_V j\omega \left(\frac{\mu\mathbf{H} \cdot \mathbf{H}^* - \epsilon\mathbf{E} \cdot \mathbf{E}^*}{2} \right) dV + \iiint_V \frac{\mathbf{E} \cdot \mathbf{J}^*}{2} dV$$

$$= 2j\omega \iiint_V \left(\frac{\mu\mathbf{H} \cdot \mathbf{H}^*}{4} - \frac{\epsilon\mathbf{E} \cdot \mathbf{E}^*}{4} \right) dV + \iiint_V \frac{\mathbf{E} \cdot \mathbf{J}^*}{2} dV$$

磁気エネルギー
magnetic energy

電気エネルギー
electric energy

オーミック・ロス
(ohmic loss)



Physical Interpretation

The physical meanings of each term in the surface integration of Poynting vector are interpreted in terms of Complex power.

この各項の持つ意味を回路における複素電力の関係により類推する。

$$V = \left(R + j\omega L + \frac{1}{j\omega C} \right) I$$

$$P = \frac{VI^*}{2} = \frac{RII^*}{2} + 2j\omega \left(\frac{LII^*}{4} - \frac{II^*}{4\omega^2 C} \right)$$

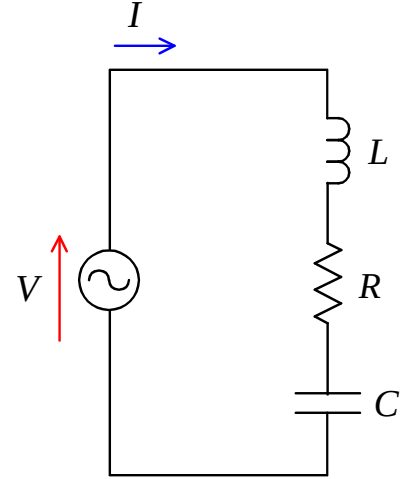
$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{H}^*}{2} \Rightarrow \text{Poynting Vector } \left[\text{Watt}/\text{m}^2 \right]$$

$$\begin{aligned} \iiint_V \frac{\mathbf{E} \cdot \mathbf{J}^*}{2} dv &\leftrightarrow \frac{RII^*}{2} \\ \text{対応関係: } \iiint_V \frac{\mu \mathbf{H} \cdot \mathbf{H}^*}{4} dv &\leftrightarrow \frac{LII^*}{4} \\ -\iiint_V \frac{\epsilon \mathbf{E} \cdot \mathbf{E}^*}{4} dv &\leftrightarrow -\frac{II^*}{4\omega^2 C} \end{aligned}$$

$$W_e = \frac{R_e}{4} (\mathbf{D} \cdot \mathbf{E}^*) \quad , \quad W_m = \frac{R_e}{4} (\mathbf{B} \cdot \mathbf{H}^*)$$

$$W_e = \frac{QV}{2} = \frac{CV^2}{2} = \frac{Q^2}{2C} = \frac{1}{2C} \left(\frac{I}{j\omega} \right)^2 = -\frac{I^2}{2\omega^2 C} \quad \text{実効値: } \frac{II^*}{4\omega^2 C}$$

$$\therefore Q = CV \quad , \quad \frac{\partial}{\partial t} Q = I \Rightarrow j\omega Q = I$$



Comparing these with the surface integration of Poynting vector, we can find the analogy between each term. We can understand that the surface integration of

Poynting vector over closed surface S $\iint_S \frac{\mathbf{E} \times \mathbf{H}^*}{2} \cdot \hat{\mathbf{n}}' dS$ represents the complex power (Watt) provided into the volume V .

Moreover, it suggests that $\frac{\mathbf{E} \times \mathbf{H}^*}{2}$ represents the complex power (vector) per unit area, though it is not always true. In many engineering problems, this assumption works well. The real part of the Poynting vector indicates the effective power flow along $\mathbf{S}$ per unit area. $\left[\text{Watt}/\text{m}^2 \right]$

これらの対応から $\iint_S \frac{\mathbf{E} \times \mathbf{H}^*}{2} \cdot \hat{\mathbf{n}}' dS$ は V 内へ供給される複素電力であることが分かる。

類推： $\frac{\mathbf{E} \times \mathbf{H}^*}{2}$ は単位面積当りの複素電力流を表す。

特にその実部は単位面積当り 1 秒間に通過するエネルギー。

Radiation power from a small dipole

Calculation using Poynting vectors

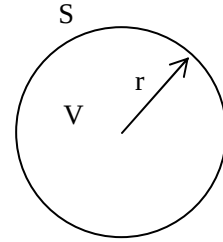
Poynting Vector を用いた微小ダイポールからの放射エネルギーの計算例

◆ Exact calculation valid everywhere 遠方界近似せず厳密に計算した場合

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{H}^*}{2}$$

Energy flowing out of the sphere r is calculated.

半径 r の球を考え、外向きへのエネルギー流を積分する。



$$P = \text{Real} \left[\int \frac{\mathbf{E} \times \mathbf{H}^*}{2} \cdot d\mathbf{s} \right]$$

$$E_r = \frac{\eta(I)}{2\pi r^2} \left(1 + \frac{1}{jkr} \right) \cos \theta \cdot e^{-jkr} \quad E_\phi = 0$$

$$E_\theta = j \frac{\eta(I)}{2\lambda r} \left(1 + \frac{1}{jkr} - \frac{1}{k^2 r^2} \right) \sin \theta \cdot e^{-jkr} \quad H_r = 0$$

$$H_\phi = j \frac{(I)}{2\lambda r} \left(1 + \frac{1}{jkr} \right) \sin \theta \cdot e^{-jkr} \quad H_\theta = 0$$

where $\eta = \sqrt{\mu/\epsilon}$ で Free space 自由空間では $120\pi[\Omega]$

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \mathbf{E} \times \mathbf{H}^* \cdot \hat{r} ds = \frac{1}{2} E_\theta \underline{H_\phi^*} ds = \left(\frac{I}{2\lambda r} \right)^2 \eta \sin^2 \theta \left(1 - \frac{1}{k^2 r^2} + \frac{1}{jkr} \right) \times \left(1 - \frac{1}{jkr} \right) ds$$

(Far field: 遠方では $H_\phi = \frac{E_\theta}{\eta}$)

Real part は

$$\frac{1}{2} \left(\frac{Il}{2\lambda r} \right)^2 \eta \sin^2 \theta \left(1 - \frac{1}{k^2 r^2} + \frac{1}{k^2 r^2} \right) ds = \frac{1}{2} \left(\frac{Il}{2\lambda r} \right)^2 \eta \sin^2 \theta ds$$

$$ds = r^2 \sin \theta d\theta d\phi$$

$$\begin{aligned} \bullet P &= \frac{1}{2} \left(\frac{Il}{2\lambda} \right)^2 \eta \int \sin^3 \theta d\theta d\phi & \int \sin^3 \theta d\theta &= \frac{4}{3} \\ &= \frac{1}{6} \left(\frac{Il}{\lambda} \right)^2 \eta \cdot 2\pi = \frac{\eta (Il)^2}{3\lambda^2} \pi \quad [\text{watt}] \end{aligned}$$

Total radiated power 全放射電力

正面方向への輻射

$$\bullet \frac{E_\theta H_\phi^*}{2} ds \Big|_{\theta=\frac{\pi}{2}} \times 4\pi r_0^2 = \frac{\eta}{2} \left(\frac{Il}{2\lambda r} \right)^2 \times 4\pi r^2 = \frac{\eta (Il)^2}{2\lambda^2} \pi$$

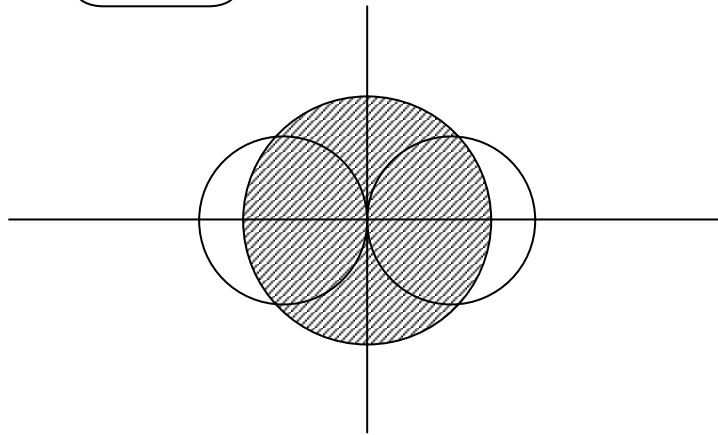
eirp effective isotropically radiated power 等価等方輻射電力

距離によらず (エネルギー保存則)

Gain at $\theta = 0^\circ$

$$\text{利得}_{\theta=\frac{\pi}{2}} = \frac{3}{2} = 1.5$$

1.76 dB



In the far field 遠方では $H_\phi = \frac{E_\theta}{\eta}$ より、

$$\left| \frac{\mathbf{E} \times \mathbf{H}^*}{2} \right| = \frac{|E_\theta|^2}{2\eta} \quad \text{と同じ}$$

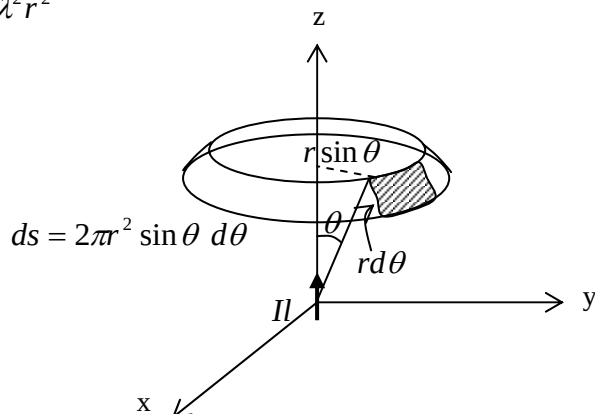
◆ **Approximate calculation valid only for Far Fields (最初から遠方界近似を用いた場合)**

微小ダイポール

$kr \gg 1$ の時

$$\left. \begin{aligned} \mathbf{E} &= E_{\theta} \hat{\theta} = \hat{\theta} j \frac{\eta}{2\lambda} \frac{Il}{r} e^{-jkr} \sin \theta \\ \mathbf{H} &= H_{\phi} \hat{\phi} = \hat{\phi} \frac{E_{\theta}}{\eta} \end{aligned} \right\} \quad \text{Far field approx. 遠方界近似}$$

$$\frac{\mathbf{E} \times \mathbf{H}^*}{2} = \hat{r} \frac{|E_{\theta}|^2}{2\eta} = \frac{\eta I^2 l^2}{8\lambda^2 r^2} \sin^2 \theta \hat{r}$$



$$\begin{aligned} \oint \frac{\mathbf{E} \times \mathbf{H}^*}{2} \cdot d\mathbf{s} &= \int_{\theta=0}^{\pi} \frac{\eta I^2 l^2}{8\lambda^2 r^2} \sin^2 \theta \cdot r^2 \sin \theta d\theta \cdot 2\pi \\ &= \frac{\pi}{4} \eta \frac{I^2 l^2}{\lambda^2} \int_0^{\pi} \sin^3 \theta d\theta \\ &\quad \frac{4}{3} \\ &= \frac{\eta \pi (Il)^2}{3\lambda^2} \quad [\text{watt}] \end{aligned}$$

同じ結果 (Identical result)