Derivation of the far field expression in terms of the normal component of the vector potential

The electric field may be expressed in a general form as:

$$\boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu} \tag{1}$$

where $\boldsymbol{A}$ is know as the vector potential, defined as:

$$\boldsymbol{A} = \frac{\mu}{4\pi}\int \boldsymbol{J}\,\frac{e^{-jkr}}{r}\,dv \tag{2}$$

Then solution of the electric field is reduced to the calculation of the current distribution.

The expression in (1) may be complicated for the calculation and also difficult to understand in its physical essence. Under some situations or physical conditions (1) may be reduced to a simpler form and this is the objective which is resumed as:

$$\boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu} \Rightarrow -j\omega\boldsymbol{A}_{\perp} \approx -j\omega(\boldsymbol{A}_{\theta}\hat{\theta} + \boldsymbol{A}_{\phi}\hat{\phi}) \tag{3}$$

The electric field $\boldsymbol{E}$ produced by current distribution may be expressed as the normal component to the propagation direction of the vector potential $\boldsymbol{A}$. The condition for the validation of (3) is:

$$r \gg 1 \tag{4.1}$$

$$kr \gg 1 \tag{4.2}$$

$$r \gg D \tag{4.3}$$

As first step in the derivation, we have, by (4.2):

$$
\begin{aligned}
&\psi \approx \frac{e^{-jkr}}{r} \\
&\nabla \psi = -\frac{1+jkr}{r^2} e^{-jkr} \boldsymbol{r} \approx -jk\psi \boldsymbol{r} \\
&\nabla \frac{\psi}{r} = -\frac{jkr+2}{r^3} e^{-jkr} \boldsymbol{r} \approx -jk\frac{\psi}{r} \boldsymbol{r}
\end{aligned} \tag{5}
$$

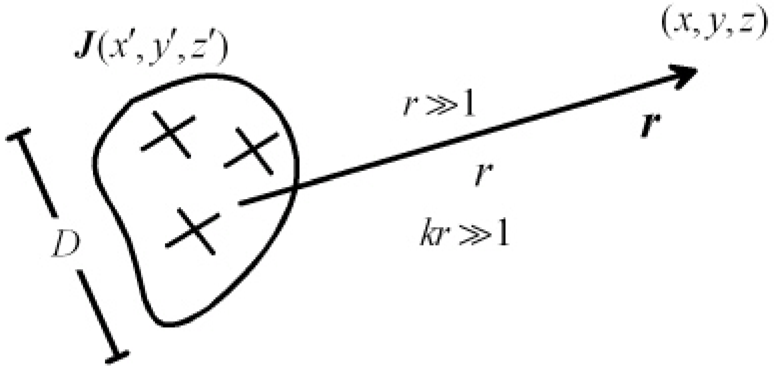

Fig. 1 Representation for the condition $r \gg 1$

Approximations in (5) are according with the Fig.1. The observation point is far from as, $r \gg 1$, the current distribution. The source distribution depends on (x', y', z') and the observation point is related with (x, y, z) .

$$
\begin{aligned}
\nabla \cdot \boldsymbol{A} &= \frac{\mu}{4\pi}\int \nabla(\boldsymbol{J}\psi)dv = \frac{\mu}{4\pi}\int (\psi\nabla\cdot\boldsymbol{J} + \boldsymbol{J}\cdot\nabla\psi)dv \\
&= -jk\frac{\mu}{4\pi}\int \boldsymbol{J}\cdot\boldsymbol{r}\psi dv = -jk\frac{\mu}{4\pi}\int J_{x'}\frac{x\psi}{r} + J_{y'}\frac{y\psi}{r} + J_{z'}\frac{z\psi}{r} dv \\
&\qquad\qquad = -jk\frac{\mu}{4\pi}\int (J_{x'}x + J_{y'}y + J_{z'}z)\frac{\psi}{r} dv
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\nabla\nabla\cdot\boldsymbol{A} = -jk\frac{\mu}{4\pi}\int & J_{x'}\frac{\psi}{r} + (J_{x'}x + J_{y'}y + J_{z'}z)\frac{\partial}{\partial x}\frac{\psi}{r} dv\ \hat{x} \\
& J_{y'}\frac{\psi}{r} + (J_{x'}x + J_{y'}y + J_{z'}z)\frac{\partial}{\partial y}\frac{\psi}{r} \quad \hat{y} \\
& J_{z'}\frac{\psi}{r} + (J_{x'}x + J_{y'}y + J_{z'}z)\frac{\partial}{\partial z}\frac{\psi}{r} \quad \hat{z}
\end{aligned}
$$

$$
\begin{aligned}
&= -jk\frac{\mu}{4\pi}\int \boldsymbol{J}\cdot\frac{\psi}{r} + (\boldsymbol{J}\cdot\boldsymbol{r}r)\nabla\frac{\psi}{r}dv \\
&= -jk\frac{\mu}{4\pi}\int \boldsymbol{J}\cdot\frac{\psi}{r} - jk\psi\boldsymbol{r}(\boldsymbol{J}\cdot\boldsymbol{r})dv
\end{aligned}
\tag{7}
$$

Now with (7) in (1) and with the conditions (4.1) and (4.3) we have:

$$
\begin{aligned}
\boldsymbol{E} &= -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu} = -j\omega\left[\boldsymbol{A} + \frac{1}{k^2}\nabla\nabla\cdot\boldsymbol{A}\right] \\
&= -j\omega\left[\boldsymbol{A} + \frac{1}{k^2}\left(-\frac{jk}{r}\boldsymbol{A} - k^2(\boldsymbol{A}\cdot\boldsymbol{r})\boldsymbol{r}\right)\right] \\
&= -j\omega\left[\boldsymbol{A} - \boldsymbol{r}(\boldsymbol{A}\cdot\boldsymbol{r})\right] = \boldsymbol{A}_{\perp}
\end{aligned}
\tag{8}
$$

$$
\boldsymbol{A}_{\perp} \approx -j\omega(\boldsymbol{A}_{\theta}\hat{\theta} + \boldsymbol{A}_{\phi}\hat{\phi})
$$

where $\boldsymbol{A}_{\perp}$ is the normal component of the vector potential to the propagation of the energy, see Fig.2

Then the relations between $\boldsymbol{E}$ and $\boldsymbol{H}$ are shown as:

$$
\begin{aligned}
\nabla\times\boldsymbol{A} &= \frac{\mu}{4\pi}\iiint_v \nabla\times(\boldsymbol{J}\psi)dv \\
&= \frac{\mu}{4\pi}\iiint_v \left[\psi\nabla\times\boldsymbol{J} + \nabla\psi\times\boldsymbol{J}\right]dv \\
&= \frac{\mu}{4\pi}\iiint_v (-jk\psi\boldsymbol{r})\times\boldsymbol{J}dv \\
&= -jk\boldsymbol{r}\times\boldsymbol{A}
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\boldsymbol{H} &= \frac{\nabla\times\boldsymbol{A}}{\mu} = \frac{jk}{\mu}\boldsymbol{r}\times\boldsymbol{A} = -\frac{jk}{\mu}\boldsymbol{r}\times\left[\boldsymbol{A} - \boldsymbol{r}(\boldsymbol{A}\cdot\boldsymbol{r})\right] \\
&= -\frac{jk}{\mu}\boldsymbol{r}\times\frac{\boldsymbol{E}}{-j\omega} = \frac{1}{\eta}\boldsymbol{r}\times\boldsymbol{E}
\end{aligned}
\tag{10}
$$

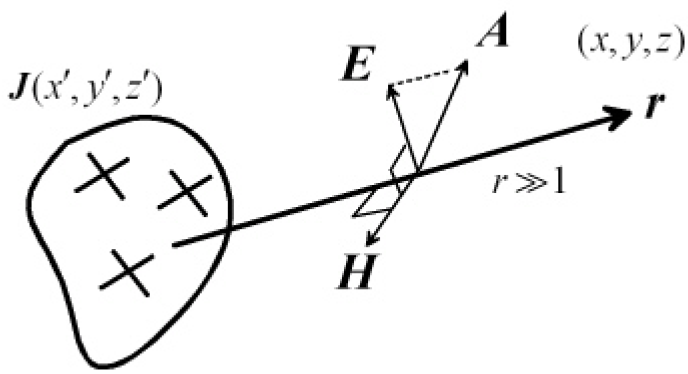

Fig. 2 Identification of the field components

Then, the electric and magnetic vectors of a field satisfying the condition in (4) are orthogonal to the direction of propagation and to each other.

$$
\begin{aligned}
&1)\ \boldsymbol{A} = \frac{\mu}{4\pi}\int \boldsymbol{J}\,\frac{e^{-jkr}}{r}\,dv \\
&2)\ \boldsymbol{E} = \boldsymbol{A}_{\perp} \\
&3)\ \boldsymbol{H} = \frac{1}{\eta}\boldsymbol{r}\times\boldsymbol{E}
\end{aligned}
\tag{11}
$$

Optional:

1- In (6) and (9) there are two terms (one in each identity), may you explain why these terms are neglected?

2- Try to read the chapter: “Plane waves in unbounded, isotropic media” of the book Stratton. Is there any difference between the derivation presented here and the one established by Stratton?

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