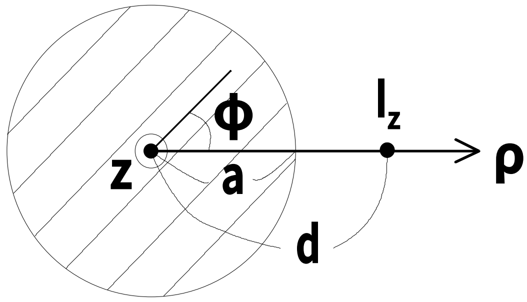
Electric Line Source Scattering by a Conducting Circular Cylinder



Analysis Methods based on

1. Huygens' principle
2. Ampere's law
3. Reciprocity Theorem

2D-problem

structure & source

$\cdots$ uniform & infinite along the z-axis $\left(\frac{\partial}{\partial z} = 0\right)$

Electric Current Source

$\rightarrow$TM - problem (H_z =0)

$$\mathbf{A} = \hat{\mathbf{z}}\Psi_A$$

$$E_\rho = E_\varphi = 0 \quad H_z = 0$$

$$E_z = \frac{k^2}{j\omega\varepsilon}\Psi_A \quad , \quad H_\rho = \frac{1}{\rho}\frac{\partial\Psi_A}{\partial\phi} = \frac{j\omega\varepsilon}{k^2}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} \quad , \quad H_\phi = -\frac{\partial\Psi_A}{\partial\rho} = -\frac{j\omega\varepsilon}{k^2}\frac{\partial E_z}{\partial\rho}$$

$E_z(\propto\Psi_A)$ is an even function of ϕ

Analysis based on Huygens' principle

Solution = Particular Solution (i.e. field due to the line current)

+

Complementary Solution

(to satisfy the boundary condition)

Boundary condition

$$E_z = 0 \quad at \quad \rho = a$$

Field produced by the line current

$$\mathbf{I} = \hat{\mathbf{z}}\, I_z$$

$$\mathbf{A} = \hat{\mathbf{z}}\left[C_1\, H_m^{(1)}(k_\rho \rho) + D_1\, H_m^{(2)}(k_\rho \rho)\right]$$

$$\times\left[C_2 \cos\left(m\phi\right) + D_2 \sin\left(m\phi\right)\right]$$

$$\times\left[A_3\, e^{-jk_z z} + B_3\, e^{+jk_z z}\right]$$

$$= \hat{\mathbf{z}}\, A_0\, H_0^{(2)}(k\rho)$$

$$\left(\because \frac{\partial}{\partial z} = 0 \Rightarrow k_z = 0\ ,\ k_\rho = k\right)$$

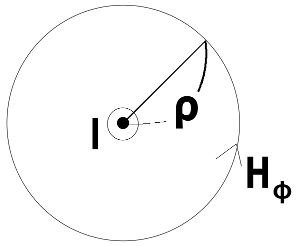
$$\left(\because \frac{\partial}{\partial \phi} = 0 \Rightarrow m = 0\ ,\ D_2 = 0\right)$$

$$\left(\because outgoing\ wave \Rightarrow C_1 = 0\right)$$

A_0 : *unknown*

$$\mathbf{A} = \hat{\mathbf{z}}\, A_0\, H_0^{(2)}(k\rho) \rightarrow E_z = -j\omega\mu\, A_0\, H_0^{(2)}(k\rho)$$

$$H_\phi = -k\, A_0 H_0^{'(2)}(k\rho) = A_0 k\, H_1^{(2)}(k\rho)$$



Ampere's Law

$$I = \lim_{\rho \to 0} \int_C \mathbf{H} \cdot d\,\mathbf{l} = \lim_{\rho \to 0} \int_0^{2\pi} H_\phi \cdot \rho \, d\phi$$

Asymptotic Expansions of Hankel function for small argument

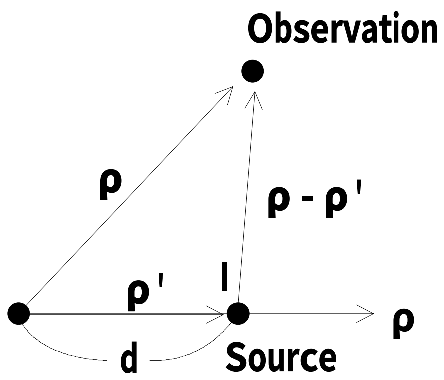
$$H_1^{(2)}(k\rho) = J_1(k\rho) - j\,N_1(k\rho) \underset{k\rho \to 0}{\simeq} \frac{k\rho}{2} + j\frac{2}{\pi}\frac{1}{k\rho} \simeq j\frac{2}{\pi}\frac{1}{k\rho}$$

$$I \simeq A_0 \; jk\,\frac{2}{\pi}\frac{1}{k\rho}\,\rho\,2\pi = 4j\,A_0$$

$$\therefore A_0 = -\frac{j}{4} I$$

$$\mathbf{A} = -\hat{\mathbf{z}}\,\frac{j}{4}\,I\,H_0^{(2)}(k\rho) \;\rightarrow\; E_z = -I\,\frac{\omega\mu}{4} H_0^{(2)}(k\rho)$$

Translation of cylindrical coordinate origin



$$E_{zs} = -I\,\frac{\omega\mu}{4} H_0^{(2)}\left(k\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)$$

$$= \begin{cases} -I\,\dfrac{\omega\mu}{4}\displaystyle\sum_{n=0}^{\infty}\varepsilon_{\mathrm{n}}\; H_n^{(2)}(kd)\; J_n(k\rho)\cos n\varphi & (\rho < d) \\ -I\,\dfrac{\omega\mu}{4}\displaystyle\sum_{n=0}^{\infty}\varepsilon_{\mathrm{n}}\; J_n^{(2)}(kd)\; H_n(k\rho)\cos n\varphi & (\rho > d) \end{cases}$$

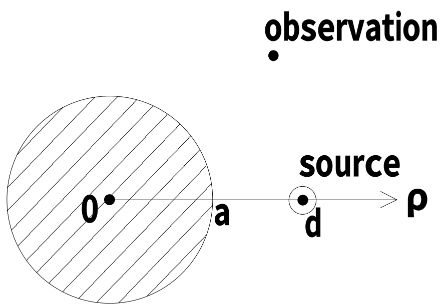
$$(\ \varepsilon_n = 1\,(n=0)\,,\ 2\,(n \neq 0)\ )$$

Complementary Solution

$$E_{zc} = \sum_{n=0}^{\infty} C_n\; \underline{H_n^{(2)}(k\rho)}\; \underline{\cos n\varphi}$$

unknown outgoing wave even function in ϕ

Boundary Condition



$$E_z = 0 \;\; at \;\; \rho = a$$

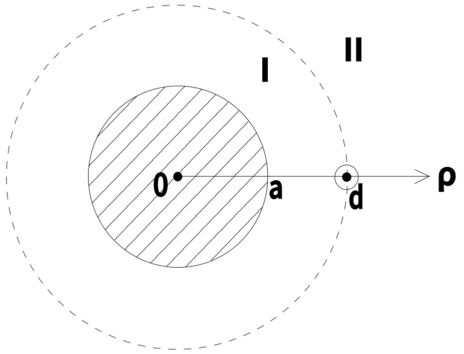
$$E_z = E_{zs} + E_{zc} = -I\,\frac{\omega\mu}{4}\sum_{n=0}^{\infty}\varepsilon_n\; H_n^{(2)}(kd)\; J_n(ka)\cos n\varphi$$

$$+\sum_{n=0}^{\infty} C_n\; H_n^{(2)}(ka)\cos n\varphi = 0$$

$$\int_0^{2\pi} \bigcirc \times \cos m\varphi a\, d\varphi \quad \Rightarrow \quad C_n = -I\,\frac{\omega\mu}{4}\,\varepsilon_n\left\{ -H_n^{(2)}(kd)\,\frac{J_n(ka)}{H_n^{(2)}(ka)} \right\}$$

$$E_z = \begin{cases} -I\,\dfrac{\omega\mu}{4}\displaystyle\sum_{n=0}^{\infty}\varepsilon_n H_n^{(2)}(kd)\left\{ J_n(k\rho) - \dfrac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right\}\cos n\varphi & (\rho < d) \\ -I\,\dfrac{\omega\mu}{4}\displaystyle\sum_{n=0}^{\infty}\varepsilon_n H_n^{(2)}(k\rho)\left\{ J_n(kd) - \dfrac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(kd) \right\}\cos n\varphi & (\rho > d) \end{cases}$$

Analysis based on Ampere's Laws



① Division of the analysis region on the source boundary (i.e. ρ = d)

Region Ⅰ : $a < \rho < d$
Region Ⅱ : $d < \rho$

② Field Expansion in each region to satisfy the boundary condition

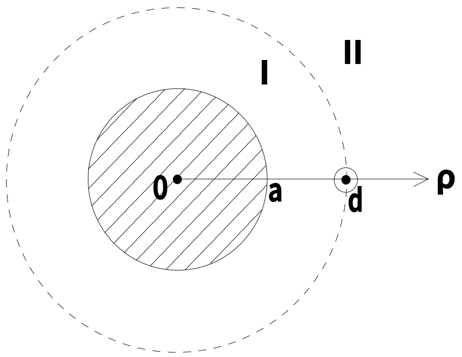
Region Ⅰ : $E_z = 0 \quad at \quad \rho = a$

Region Ⅱ : $E_z, H_\varphi \to 0 \quad for \quad \rho \to \infty$

③ Field continuity condition on the source boundary

E_z : *continuous*
H_φ : *discontinuity at the source*

Field Expansion of Each Region



Region Ⅰ

$$E_{z_{\mathrm{I}}} = \sum_{n=0}^{\infty} \left\{ A_n \, J_n(k\rho) + B_n \, H_n^{(2)}(k\rho) \right\} \cos n\phi$$

A_n , B_n : *unknown*

Boundary Condition : $E_z = 0$ *at* $\rho = a$ *for all* ϕ

$$A_n \, J_n(ka) + B_n \, H_n^{(2)}(ka) = 0 \quad \left(\because E_z = \frac{k^2}{j\omega\varepsilon} \Psi_A \right)$$

$$\therefore B_n = -\frac{J_n(ka)}{H_n^{(2)}(ka)} A_n$$

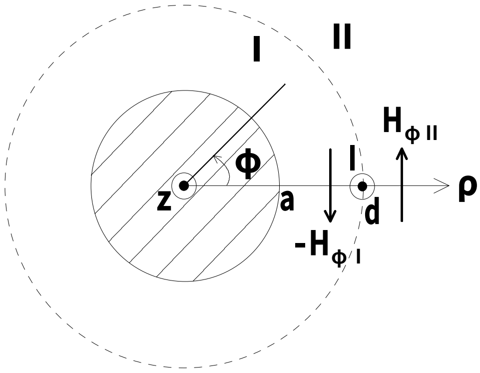
$$\therefore E_{z_{\mathrm{I}}} = \sum_{n=0}^{\infty} A_n \left\{ J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right\} \cos n\phi$$

Region Ⅱ

$$E_{z_{\mathrm{II}}} = \sum_{n=0}^{\infty} C_n \, \underline{H_n^{(2)}(k\rho)} \cos n\phi$$

C_n → unknown; $H_n^{(2)}(k\rho)$ → Radiation condition

Field Continuity Condition on the source boundary (ρ = d)



$$E_{z_{\mathrm{I}}} = E_{z_{\mathrm{II}}} \qquad \cdots A$$

$$H_{\phi_{\mathrm{II}}} - H_{\phi_{\mathrm{I}}} = I\ \delta(\phi) \quad \cdots B$$

$$E_{z_{\mathrm{I}}} = \sum_{n=0}^{\infty} A_n \left\{ J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right\} \cos n\phi$$

$$E_{z_{\mathrm{II}}} = \sum_{n=0}^{\infty} C_n\ H_n^{(2)}(k\rho) \cos n\phi$$

$$H_{\phi_{\mathrm{I}}} = -\frac{j\omega\varepsilon}{k} \sum_{n=0}^{\infty} A_n \left\{ J_n'(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n'^{(2)}(k\rho) \right\} \cos n\phi$$

$$H_{\phi_{\mathrm{II}}} = -\frac{j\omega\varepsilon}{k} \sum_{n=0}^{\infty} C_n\ H_n'^{(2)}(k\rho) \cos n\phi$$

From A

$$A_n \left\{ J_n(kd) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(kd) \right\} = C_n\ H_n^{(2)}(kd) \quad \cdots A'$$

From B

$$-\frac{j\omega\varepsilon}{k} \left[\sum_{n=0}^{\infty} C_n H_n'^{(2)}(kd) \cos n\phi - \sum_{n=0}^{\infty} A_n \left\{ J_n'(kd) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n'^{(2)}(kd) \right\} \cos n\phi \right]$$

$$= I\ \delta(\phi) \quad \cdots B'$$

$$\int_{-\pi}^{\pi} \bigcirc \times \cos m\phi \, d\phi$$

$$-\frac{j\omega\varepsilon}{k}\frac{2\pi d}{\varepsilon_n}\left[C_n \, H_n'^{(2)}(kd) - A_n \left\{ J_n'(kd) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n'^{(2)}(kd) \right\} \right] = I \quad \cdots B''$$

$$\left(\because \int_{-\pi}^{\pi} \cos n\phi \cos m\phi \, d\phi = \frac{2\pi}{\varepsilon_n}\delta_{nm} \right) \longrightarrow \text{ kroneker's delta}$$

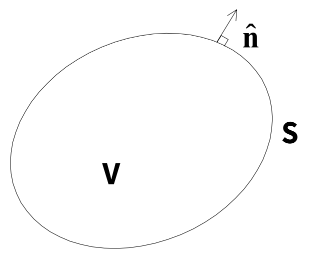
From A', B''

$$A_n = \frac{\varepsilon_n \, j\omega\mu}{2\pi kd \left\{ \frac{J_n(kd)}{H_n^{(2)}(kd)} \cdot H_n'^{(2)}(kd) - J_n'(kd) \right\}} = -\frac{\omega\mu}{4} I \, \varepsilon_n \, H_n^{(2)}(kd)$$

$$\left(\because J_n(k\rho) H_n'^{(2)}(k\rho) - J_n'(k\rho) H_n^{(2)}(k\rho) = -\frac{2j}{\pi k\rho} \right)$$

$$C_n = -\frac{\omega\mu}{4} I \, \varepsilon_n \left(J_n(kd) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(kd) \right)$$

Reciprocity Theorem



Two sets of source and its field

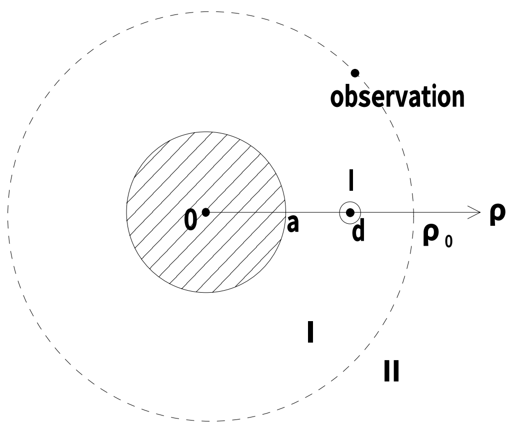
Source		field
$(\mathbf{J}_1, \mathbf{M}_1)$	$\rightarrow$	$(\mathbf{E}_1, \mathbf{H}_1)$
$(\mathbf{J}_2, \mathbf{M}_2)$	$\rightarrow$	$(\mathbf{E}_2, \mathbf{H}_2)$

satisfy the following equation

$$\int_S \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) \cdot \hat{\mathbf{n}}\, dS$$
$$= \int_V \left\{\left(\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2\right) - \left(\mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2\right)\right\} dV$$

$\hat{\mathbf{n}}$: outward normal unit vector

Analysis based on Reciprocity Theorem



① Division of the analysis region on the observation boundary $(\rho = \rho_0)$

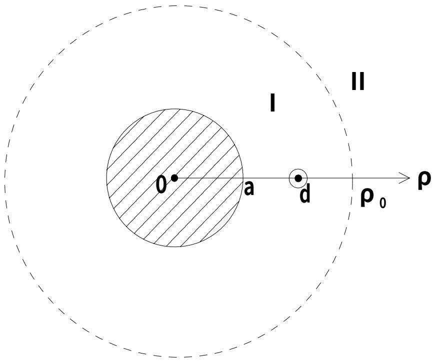
Region Ⅰ ： $a < \rho < \rho_0$
Region Ⅱ ： $\rho_0 < \rho$

② Apply the Reciprocity Theorem to each region

$\mathbf{E}_1$, $\mathbf{H}_1$, $\mathbf{J}_1$, $\mathbf{M}_1$: interested

$\mathbf{E}_2$, $\mathbf{H}_2$, $\mathbf{J}_2$, $\mathbf{M}_2$: auxiliary

$\rho_0 > d$



Apply to the Region Ⅰ

$\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}}$ (outward)
$\mathbf{J_2} = \mathbf{M_2} = \mathbf{0}$ (outside of Region Ⅰ)

$$E_{2z} = \left\{ J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right\} \cos n\phi$$

$$H_{2\phi} = -\frac{j\omega\varepsilon}{k} \left\{ J_n'(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n'^{(2)}(k\rho) \right\} \cos n\phi$$

(to satisfy the boundary condition)

$$\mathbf{J}_1 = \hat{\mathbf{z}}\, I\, \delta(\rho - d)\, \delta(\phi)\,, \quad \mathbf{M}_1 = \mathbf{0}\,, \quad E_{1z}\,, \quad H_{1\phi}$$

$$\int_0^{2\pi} (-E_{1z}\cdot H_{2\phi} + E_{2z}\cdot H_{1\phi})\cdot \rho_0\, d\phi = \int_V E_{2z}\cdot J_{1z}\, dV$$

$$\int_0^{2\pi}\left[-E_{1z}\left\{-\frac{j\omega\varepsilon}{k}\left(J_n'(k\rho_0) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n'^{(2)}(k\rho_0)\right)\right\}\cos n\phi \right.$$
$$\left. + \left(J_n(k\rho_0) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho_0)\right)\cos n\phi \cdot H_{1\phi}\right]\rho_0\, d\phi$$
$$= I\left(J_n(k\rho_0) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(kd)\right) \quad \cdots A$$

Apply to the Region Ⅱ

$\hat{\mathbf{n}} = -\hat{\boldsymbol{\rho}}$, $\mathbf{J}_2 = \mathbf{M}_2 = \mathbf{0}$ (outside of Region Ⅱ)

$$\left.\begin{aligned} E_{2z} &= H_n^{(2)}(k\rho)\cos n\phi \\ H_{2\phi} &= -\frac{j\omega\varepsilon}{k} H_n'^{(2)}(k\rho)\cos n\phi \end{aligned}\right\} \textit{ to satisfy the radiation condition}$$

$\mathbf{J}_1 = \mathbf{0}$ (*located in* Region Ⅰ), $\mathbf{M}_1 = \mathbf{0}$

$$-\int_0^{2\pi}\left[-E_{1z}\left(-\frac{j\omega\varepsilon}{k} H_n'^{(2)}(k\rho_0)\cos n\phi\right) + H_n^{(2)}(k\rho_0)\cos n\phi \cdot H_{1\phi}\right]\rho_0\, d\phi = 0 \quad \cdots B$$

From A , B (to eliminate $H_{1\phi}$)

$$\int_0^{2\pi} E_{1z}\cdot \cos n\phi\, d\phi = -\frac{\pi\omega\mu}{2} I\left(J_n(kd) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(kd)\right) H_n^{(2)}(k\rho_0)$$

Substituting $E_{1z} = \sum_{m=0}^{\infty} C_m \cos m\phi$

$$C_m = -\frac{\omega\mu}{4}\varepsilon_m I\left(J_m(kd) - \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)}(kd)\right) H_m^{(2)}(k\rho_0)$$