Fields in cylindrical coordinates

$\mathbf{A} = \hat{\mathbf{z}}\ \Psi_A \qquad \mathbf{F} = \hat{\mathbf{z}}\ \Psi_F$

$$E_\rho = \frac{1}{\sigma + j\omega\varepsilon}\frac{\partial^2 \Psi_A}{\partial\rho\,\partial z} - \frac{1}{\rho}\frac{\partial \Psi_F}{\partial\phi} \qquad H_\rho = \frac{1}{\rho}\frac{\partial\Psi_A}{\partial\phi} + \frac{1}{j\omega\mu}\frac{\partial^2 \Psi_F}{\partial\rho\,\partial z}$$

$$E_\phi = \frac{1}{\sigma + j\omega\varepsilon}\frac{1}{\rho}\frac{\partial^2 \Psi_A}{\partial\phi\,\partial z} + \frac{\partial \Psi_F}{\partial\rho} \qquad H_\phi = -\frac{\partial\Psi_A}{\partial\rho} + \frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial^2 \Psi_F}{\partial\phi\,\partial z}$$

$$E_x = \frac{1}{\sigma + j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\Psi_A \qquad H_z = \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\Psi_F$$

——— TM (transverse magnetic) to z

-·-·-·- TE (transverse electric) to z

Arbitrary field ⋯ sum of a TM field and a TE field

Helmholtz Equation in rectangular coordinates

$$\mathbf{A}_c = \hat{\mathbf{z}}\ \Psi_A(\rho,\phi,z) \ \rightarrow\ \nabla^2\mathbf{A}_c + k^2\mathbf{A}_c = \mathbf{0}$$

$$\nabla^2\Psi_A(\rho,\phi,z) + k^2\Psi_A(\rho,\phi,z) = 0$$

$$\begin{pmatrix} Note! \\ \nabla^2(\hat{\mathbf{z}}\ A_z) = \hat{\mathbf{z}}\nabla^2 A_z \\ \nabla^2(\hat{\boldsymbol{\rho}}\ A_\rho) \neq \hat{\boldsymbol{\rho}}\nabla^2 A_\rho \\ \nabla^2(\hat{\boldsymbol{\varphi}}\ A_\phi) \neq \hat{\boldsymbol{\varphi}}\nabla^2 A_\phi \end{pmatrix}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi_A}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Psi_A}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\Psi_A + k^2\Psi_A = 0$$

$\Psi_A(\rho,\phi,z) = R(\rho)\,\Phi(\phi)\,Z(z)$ ：separation of variables

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} + \underline{\frac{1}{Z}\frac{d^2 Z}{dz^2}} + k^2 = 0$$

independent of ρ and ϕ

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -k_z^2 \quad \text{constant}$$

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\underline{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}+(k^2-k_z^2)\,\rho^2=0$$

independent of ρ and z

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-n^2 \quad \text{constant}$$

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)-n^2+(k^2-k_z^2)\,\rho^2=0$$

$$\begin{cases} \rho\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right)+\left[(k_\rho\rho)^2-n^2\right]R=0 & \textit{Bessel equation of order n} \\ \dfrac{d^2\Phi}{d\phi^2}+n^2\Phi=0 & \textit{harmonic equation} \\ \dfrac{d^2Z}{dz^2}+k_z^2Z=0 & \textit{harmonic equation} \end{cases}$$

$$k_\rho^2+k_z^2=k^2$$

$$\Psi_{k_\rho,n,k_z}=\underbrace{B_n(k_\rho\rho)}_{R(\rho)}\;\underbrace{h(n\phi)}_{\Phi(\phi)}\;\underbrace{h(k_zz)}_{Z(z)}$$

Bessel function $\quad B_n(k_\rho\rho)\sim J_n(k_\rho\rho)\,,\,N_n(k_\rho\rho)\,,\,H_n^{(1)}(k_\rho\rho)\,,\,H_n^{(2)}(k_\rho\rho)$

harmonic function $\quad h(n\phi)\ \sim\ \sin n\phi\ ,\ \cos n\phi\ ,\ e^{+jn\phi}\ ,\ e^{-jn\phi}$

$h(k_zz)\ \sim\ \sin k_zz\ ,\ \cos k_zz\ ,\ e^{+jk_zz}\ ,\ e^{-jk_zz}$

Two of these are linearly independently

period of $2\pi \quad h(n\phi)=h\{n(\phi+2\pi)\}$

n : integer

Linear combination of wave function

$$\Psi(\rho,\phi,z)=\sum_{n}\sum_{k_\rho} C_{n,k_\rho}\Psi_{k_\rho,n,k_z}$$

$$or \qquad\qquad k_\rho \text{ or } k_z : discrete$$

$$=\sum_{n}\sum_{k_z} D_{n,k_z}\Psi_{k_\rho,n,k_z}$$

General wave functions

$$\Psi(\rho,\phi,z)=\sum_{n}\int_{k_\rho} f_n(k_\rho)\Psi_{k_\rho,n,k_z}$$

$$or \qquad\qquad k_\rho \text{ or } k_z : continuous$$

$$=\sum_{n}\int_{k_z} g_n(k_z)\Psi_{k_\rho,n,k_z}$$

Bessel functions

$$\begin{cases} J_n(k\rho)=\frac{1}{2}\left\{H_n^{(1)}(k\rho)+H_n^{(2)}(k\rho)\right\}\propto\frac{1}{\sqrt{k\rho}}\cos k\rho \\ N_n(k\rho)=\frac{1}{2j}\left\{H_n^{(1)}(k\rho)-H_n^{(2)}(k\rho)\right\}\propto\frac{1}{\sqrt{k\rho}}\sin k\rho \\ H_n^{(1)}(k\rho)=J_n(k\rho)+j\,N_n(k\rho)\propto\frac{1}{\sqrt{k\rho}}\,e^{+jk\rho} \quad (inward\ traveling) \\ H_n^{(2)}(k\rho)=J_n(k\rho)-j\,N_n(k\rho)\propto\frac{1}{\sqrt{k\rho}}\,e^{-jk\rho} \quad (outward\ traveling) \end{cases}$$

Two of them are linearly independent

Modified Bessel functions (k : imaginary)

$$\begin{cases} I_n(\alpha\rho)=j^n\,J_n(-j\alpha\rho) \qquad \propto e^{\alpha\rho} \\ K_n(\alpha\rho)=\frac{\pi}{2}(-j)^{n+1}H_n^{(2)}(-j\alpha\rho) \quad \propto e^{-\alpha\rho} \ (decaying) \end{cases}$$

Edge condition

$\Psi_{k_\rho,n,k_z} = J_n(k_\rho \rho)\, e^{jn\phi}\, e^{jk_z z}$ ρ =0 included

The field must be finite at ρ =0 $\left(\int_V \left(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right) dV \to finite\right)$

Radiation condition

$\Psi_{k_\rho,n,k_z} = H_n^{(2)}(k_\rho \rho)\, e^{jn\phi}\, e^{jk_z z}$ ρ →∞ included

The field vanish for large ρ if k_ρ is complex

The field represents the outward traveling waves if k_ρ is real

$$\lim_{r\to\infty}\left\{ r\left(\frac{\partial \Psi}{\partial r} + jk\Psi\right)\right\} = 0 \quad (for\ 3D)$$

$$\lim_{\rho\to\infty}\left\{ \sqrt{\rho}\left(\frac{\partial \Psi}{\partial \rho} + jk\Psi\right)\right\} = 0 \quad (for\ 2D)$$