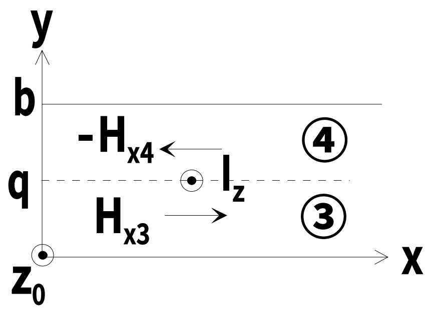
Field Continuity Condition

at y = p



$$\begin{cases} E_{z3} = E_{z4} & \cdots A' \\ H_{x3} - H_{x4} = I_z\,\delta(x-p) & \cdots B' \end{cases}$$

A'

$$\frac{k^2}{j\omega\varepsilon}\int_0^{\infty} C(k_x)\sin k_x x \sin k_y q\, dk_x = \frac{k^2}{j\omega\varepsilon}\int_0^{\infty} D(k_x)\sin k_x x\, \sin k_y (q-b)\, dk_x$$

Fourier transform $\int_0^{\infty} \bigcirc \times \sin k_x x\, dx$

$$\rightarrow \quad C(k_x)\sin k_y q = D(k_x)\sin k_y (q-b) \qquad \cdots C'$$

$$\left(\begin{array}{l} \because F_s(x) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(k)\sin kx\, dk \\ \quad \textit{For an odd function} \quad f(k) \\ f(k) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} F(x)\sin kx\, dx = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \sin kx \left[\int_0^{\infty} f(t)\sin tx\, dt\right] dx \end{array} \right)$$

B'

$$\int_0^{\infty} C(k_x)\, k_y \sin k_x x \cos k_y q\, dk_x - \int_0^{\infty} D(k_x)\, k_y \sin k_x x\, \cos k_y (q-b)\, dk_x = I_z\, \delta(x-p)$$

Fourier transform $\int_0^{\infty} \bigcirc \times \sin k_x x\, dx$

$$C(k_x)\, k_y \cos k_y q - D(k_x) k_y \cos k_y (q-b) = \frac{2}{\pi} I_z \sin k_x p \qquad \cdots D'$$

$C', D' \rightarrow$

$$C(k_x) = -\frac{2I_z}{\pi k_y}\frac{\sin k_x p\ \sin k_y (q-b)}{\sin k_y b}$$

$$D(k_x) = -\frac{2I_z}{\pi k_y}\frac{\sin k_x p\ \sin k_y q}{\sin k_y b}$$

Region③ $$\Psi_{TM} = -\frac{2I_z}{\pi}\int_0^{\infty}\frac{1}{k_y \sin k_y b}\sin k_x x \sin k_x p \sin k_y y \sin k_y (q-b)\, dk_x$$

Region④ $$\Psi_{TM} = -\frac{2I_z}{\pi}\int_0^{\infty}\frac{1}{k_y \sin k_y b}\sin k_x p \sin k_x x \sin k_y q \sin k_y (y-b)\, dk_x$$

Spectral domain expansion

$$\Psi_{TM} = -\frac{2I_z}{\pi}\int_0^{\infty}\frac{1}{k_y \sin k_y b}\sin k_x x_< \sin k_x x_> \sin k_y y_< \sin k_y (y_> -b)\, dk_x \quad \cdots A$$

$$x_< = \begin{cases} x \ (\ for\ x < p) \\ p \ (\ for\ x > p) \end{cases} \quad x_> = \begin{cases} p \ (\ for\ x < p) \\ x \ (\ for\ x > p) \end{cases} \quad y_< = \begin{cases} y \ (\ for\ y < q) \\ q \ (\ for\ y > q) \end{cases} \quad y_> = \begin{cases} q \ (\ for\ y < q) \\ y \ (\ for\ y > q) \end{cases}$$

$A \cdots even\ function\ for\ k_x$

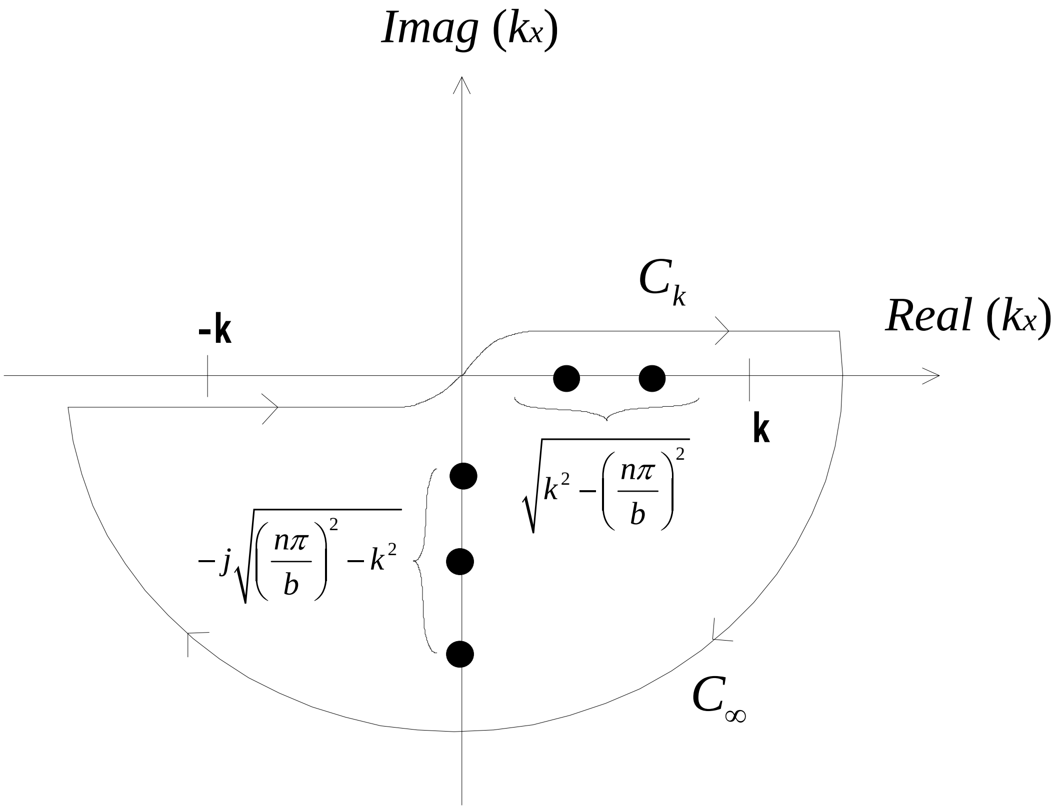
$$\begin{aligned}
\Psi_{TM} &= -\frac{I_z}{\pi}\int_{-\infty}^{\infty}\frac{1}{k_y \sin k_y b}\sin k_x x_< \sin k_x x_> \sin k_y y_< \sin k_y (y_> -b)\, dk_x \\
&= -\frac{I_z}{\pi}\int_{-\infty}^{\infty}\frac{\sin k_y y_< \sin k_y (y_> -b)}{k_y \sin k_y b}\ \frac{e^{+jk_x x_>} - e^{-jk_x x_>}}{2j}\sin k_x x_<\, dk_x \\
&= \frac{I_z}{2j\pi}\int_{-\infty}^{\infty}\frac{\sin k_y y_< \sin k_y (y_> -b)}{k_y \sin k_y b}\ e^{-jk_x x_>}\ \sin k_x x_<\, dk_x \\
&\qquad - \frac{I_z}{2j\pi}\int_{-\infty}^{\infty}\frac{\sin k_y y_< \sin k_y (y_> -b)}{k_y \sin k_y b}\ e^{+jk_x x_>}\ \sin k_x x_<\, dk_x \\
&= \frac{I_z}{j\pi}\int_{-\infty}^{\infty}\frac{\sin k_y y_< \sin k_y (y_> -b)}{k_y \sin k_y b}\ e^{-jk_x x_>}\ \sin k_x x_<\, dk_x
\end{aligned}$$

Residue Theory

denominator = 0 → $k_y = 0$ or $\sin k_y b = 0$

Integrand = finite $k_y b = n\pi$ (n : integer)

$k_x = \sqrt{k^2 - k_y^2}$ $\mathrm{Real}(k_x) \geq 0$ $\mathrm{Imag}(k_x) \leq 0$



Contribution from $C_\infty = 0$

$$\left(\begin{array}{l} \because k_x = u - jv \quad (v \to +\infty) \\ \left|Integrand\right| = \left| \dfrac{e^{vy_<}\ e^{-v(y_> - b)}}{ve^{vb}}\ e^{-vx_>}\ e^{+vx_<} \right| \to 0 \end{array} \right)$$

$$\Psi_{TM} = \frac{I_z}{j\pi}(-2\pi j)\times\sum_n \text{(Residues)}$$

$$\text{Residues} = \lim_{\substack{k_x \to k_{x_n} \\ k_y \to \frac{n\pi}{b}}} (k_x - k_{x_n}) \frac{\sin k_y y_< \; \sin k_y (y_> - b)}{k_y \sin k_y b} e^{-jk_x x_>} \sin k_x x_<$$

$$= \lim_{\substack{k_x \to k_{x_n} \\ k_y \to \frac{n\pi}{b}}} \frac{k_x - k_{x_n}}{\sin k_y b} \; \frac{\sin k_y y_< \; \sin k_y (y_> - b)}{k_y} e^{-jk_x x_>} \sin k_x x_<$$

$$= \lim_{\substack{k_x \to k_{x_n} \\ k_y \to \frac{n\pi}{b}}} \frac{\frac{d}{dk_x}(k_x - k_{x_b})}{\frac{d}{dk_x}(\sin k_y b)} \; \frac{\sin k_y y_< \; \sin k_y (y_> - b)}{k_y} e^{-jk_x x_>} \sin k_x x_<$$

$$= \frac{1}{b \cos k_{y_n} b \left(-\frac{k_{x_n}}{k_{y_n}} \right)} \; \frac{\sin k_{y_n} y_< \; \sin k_{y_n} y_> \; \cos k_{y_n} b}{k_{y_n}} e^{-jk_{x_n} x_>} \sin k_{x_n} x_<$$

$$= -\frac{1}{k_{x_n} b} \sin k_{y_n} y_< \; \sin k_{y_n} y_> \; e^{-jk_{x_n} x_>} \sin k_{x_n} x_<$$