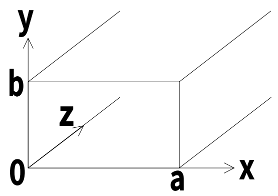
Ex. Rectangular waveguide



TM to z , traveling wave toward +z

$$\mathbf{A} = \Psi_A\,\hat{\mathbf{z}} \quad , \quad \Psi_A = h(k_x x)\,h(k_y y)\,e^{-jk_z z}$$

$$E_z = \frac{1}{j\omega\varepsilon}(k^2 - k_z^2)\,\Psi_A$$

Boundary condition： $E_z = 0 \quad at \quad x = 0\,,a \quad and \quad y = 0\,,b$

$$h(k_x x) = \sin k_x x \qquad k_x = \frac{m\pi}{a} \qquad m = 1\,,2\,,3\cdots$$

$$h(k_y y) = \sin k_y y \qquad k_y = \frac{n\pi}{b} \qquad n = 1\,,2\,,3\cdots$$

$$\Psi_A = \sin\frac{m\pi x}{a}\,\sin\frac{n\pi x}{b}\,e^{-jk_z z}$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_z^2 = k^2$$

TE to z

$$\mathbf{F} = \Psi_F\,\hat{\mathbf{z}} \quad , \quad E_x = -\frac{\partial \Psi_F}{\partial y} \quad , \quad E_y = \frac{\partial \Psi_F}{\partial x}$$

Boundary condition： $E_x = 0 \quad at \quad y = 0\,,b$

$$E_y = 0 \quad at \quad x = 0\,,a$$

$$h(k_x x) = \cos k_x x \qquad k_x = \frac{m\pi}{a} \qquad m = 1\,,2\,,3\cdots$$

$$h(k_y y) = \cos k_y y \qquad k_y = \frac{n\pi}{b} \qquad n = 1\,,2\,,3\cdots$$

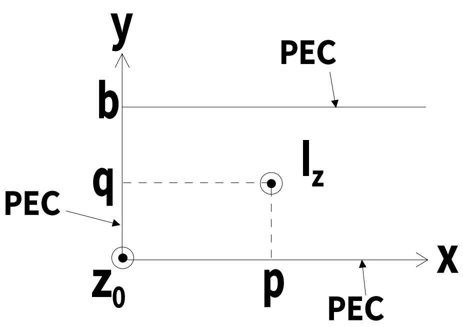
except $m = n = 0 \quad (\Rightarrow \mathbf{E} = 0\,)$

$$\Psi_A = \cos\frac{m\pi x}{a}\,\cos\frac{n\pi x}{b}\,e^{-jk_z z}$$

$$\gamma_{mn} = jk_z = \begin{cases} j\beta = j\sqrt{k^2 - k_{cmn}^2} & k > k_{cmn} \\ \alpha \ = \ \sqrt{k_{cmn}^2 - k^2} & k < k_{cmn} \end{cases}$$

$$k_{cmn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$: cut off wavenumber

Field expansion in terms of modes



Ex. Shorted parallel plates

+ line electric current I_z

2D problem

Structure and source : uniform in z

$$\rightarrow \frac{\partial}{\partial z} = 0 \quad (k_z = 0)$$

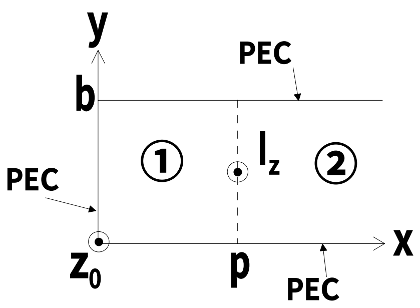
$$k_x^2 + k_y^2 = k^2$$ (One of k_x and k_y is chosen independently)

z – directed electric current I_z

$\rightarrow$ couples to E_z in the sense of inner product of vectors

$\rightarrow$ TM to z

1. k_y is chosen



$$E_z = \frac{1}{j\omega\mu}(k^2 - k_z^2)\Psi_{TM}$$ Boundary condition $\rightarrow E_z = 0\ (\ x = 0 \quad and \quad y = 0\,,\,b\ )$

Region① $\Psi_{TM} = \sum_n A_n \sin\frac{n\pi}{b}y \ \sin k_x x$

- A_n → Unknown
- $\frac{n\pi}{b}$ → k_y → $\Psi_{TM} = 0 \quad at \quad y = 0\,,\,b$ (n = 0 , 1 , 2 ⋯)
- $\sin k_x x$ → $\Psi_{TM} = 0 \quad at \quad x = 0$

Region② $\Psi_{TM} = \sum_n B_n \ \sin\frac{n\pi}{b}y \ \ e^{-jk_x x}$

- B_n → Unknown
- $\frac{n\pi}{b}$ → k_y
- $e^{-jk_x x}$ → ($\mathrm{Im}(k_x) \le 0$)

$$\Psi_{TM} \rightarrow 0 \quad for \quad x \rightarrow \infty$$

$k_y = \frac{n\pi}{b}$ ：discrete

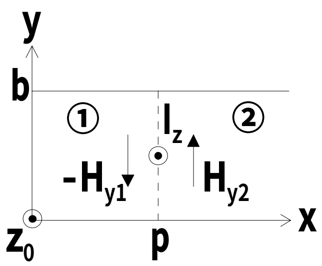
$$k_x = \begin{cases} \sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2} & \left(\frac{n\pi}{b} < k\right) \\ -j\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2} & \left(\frac{n\pi}{b} > k\right) \end{cases}$$

TM to z , $\frac{\partial}{\partial z} = 0$

$$E_x = E_y = H_z = 0$$

$$E_z = \frac{k^2}{j\omega\varepsilon}\Psi_{TM} \quad , \quad H_x = \frac{\partial \Psi_{TM}}{\partial y} \quad , \quad H_y = -\frac{\partial \Psi_{TM}}{\partial x}$$

Field Continuity Condition



At x = p

$$\begin{cases} E_{z1} = E_{z2} & \cdots A \\ -H_{y1} + H_{y2} = I_z\, \delta(y-q) & \cdots B \end{cases}$$

A $$\frac{k^2}{j\omega\varepsilon}\sum_n A_n \sin\frac{n\pi y}{b}\sin k_x p = \frac{k^2}{j\omega\varepsilon}\sum_n B_n \sin\frac{n\pi y}{b}\, e^{-jk_x p}$$

$$\int_0^b \bigcirc \times \sin\frac{m\pi y}{b}\, dy \;\rightarrow\; A_n \sin k_x p = B_n e^{-jk_x p} \qquad \cdots C$$

$$\left(\because \int_0^b \sin\frac{m\pi y}{b}\sin\frac{n\pi y}{b}\, dy = \frac{b}{2} \;\; (\, for\ m = n \neq 0\,)\;,\; 0\,(\, for\ m \neq n\,) \right)$$

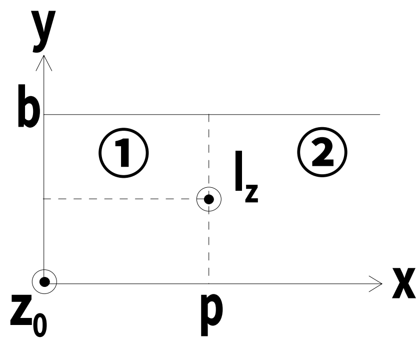
B $$-\left\{ -\sum_n A_n \sin\frac{n\pi y}{b}\; k_x \cos k_x p \right\} + \left\{ -\sum_n B_n \sin\frac{n\pi y}{b}\;(-\,jk_x)\, e^{-jk_x p} \right\} = I_z\, \delta(y-q)$$

$$\int_0^b \bigcirc \times \sin\frac{m\pi y}{b}\, dy$$

$$A_n \frac{b}{2} k_x \cos k_x p + B_n \frac{b}{2}\, jk_x\, e^{-jk_x p} = I_z \sin\frac{n\pi q}{b} \qquad \cdots D$$

C , D $\rightarrow$ $$A_n = I_z \frac{2}{k_x b}\, e^{-jk_x p} \sin\frac{n\pi q}{b}$$

$$B_n = I_z \frac{2}{k_x b} \sin k_x p \sin\frac{n\pi q}{b}$$

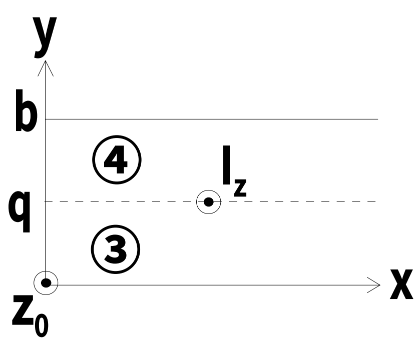


Region① $$\Psi_{TM} = I_z \sum_n \frac{2}{k_x b} \sin k_x x \; e^{-jk_x p} \; \sin \frac{n\pi}{b} y \sin \frac{n\pi q}{b}$$

Region② $$\Psi_{TM} = I_z \sum_n \frac{2}{k_x b} \sin k_x p \; e^{-jk_x x} \; \sin \frac{n\pi q}{b} \sin \frac{n\pi}{b} y$$

Source point at (p , q) field point at (x , y)

2. k_x is chosen



Region③ $$\Psi_{TM} = \int_0^\infty C(k_x) \; \underline{\sin k_x x} \; \underline{\sin k_y y} \; dk_x$$

Unknown $\quad \Psi_{TM} = 0 \quad at \quad x = 0 \qquad \Psi_{TM} = 0 \quad at \quad y = 0$

Region④ $$\Psi_{TM} = \int_0^\infty D(k_x) \; \underline{\sin k_x x} \; \underline{\sin k_y (y-b)} \; dk_x$$

Unknown $\quad \Psi_{TM} = 0 \quad at \quad x = 0 \qquad \Psi_{TM} = 0 \quad at \quad y = b$

k_x : continuous

$k_y = \sqrt{k^2 - k_x^2}$