References

・C.A.Balanis , “Advanced Engineering Electromagnetics” , John Wiley & Sons , 1989

・R.F.Harrington , “Time-Harmonic Electromagnetic Fields” , Mc-Graw Hill , 1961

・T.Sekiguchi , “Electromagnetic Waves” , Asakura , 1976 (in Japanese)

Scattering Problems

Analytically solved

Fields ⋯ eigenmode functions in coordinate systems

Ex.

Trigonometric functions ⋯ rectangular

Bessel functions ⋯ cylindrical

Lengendre functions ⋯ spherical

Numerical solved

Ex.

MoM (method of moment)

FEM (finite element method)

FDTD (finite difference time domain method)

Solutions in a homogeneous region containing sources

$\mathbf{E} = \mathbf{E}_p + \mathbf{E}_c$, $\mathbf{H} = \mathbf{H}_p + \mathbf{H}_c$

$\mathbf{E}_p, \mathbf{H}_p$: particular solution

$$\begin{cases} -\nabla \times \mathbf{E}_p = j\omega\mu \mathbf{H}_p + \mathbf{M} \\ \nabla \times \mathbf{H}_p = (\sigma + j\omega\varepsilon)\mathbf{E}_p + \mathbf{J} \end{cases}$$

$\mathbf{E}_c, \mathbf{H}_c$: complementary solution (source - free)

$$\begin{cases} -\nabla \times \mathbf{E}_c = j\omega\mu \mathbf{H}_c \\ \nabla \times \mathbf{H}_c = (\sigma + j\omega\varepsilon)\mathbf{E}_c \end{cases}$$

Particular solution

$\mathbf{E}_p = \mathbf{E}_{pA} + \mathbf{E}_{pF}$, $\mathbf{H}_p = \mathbf{H}_{pA} + \mathbf{H}_{pF}$

$\mathbf{E}_{pA}, \mathbf{H}_{pA}$: fields due to electric source $\mathbf{J}$

$$\begin{cases} -\nabla \times \mathbf{E}_{pA} = j\omega\mu \mathbf{H}_{pA} \\ \nabla \times \mathbf{H}_{pA} = (\sigma + j\omega\varepsilon)\mathbf{E}_{pA} + \mathbf{J} \end{cases}$$

$\mathbf{E}_{pF}, \mathbf{H}_{pF}$: fields due to magnetic source $\mathbf{M}$

$$\begin{cases} -\nabla \times \mathbf{E}_{pF} = j\omega\mu \mathbf{H}_{pF} + \mathbf{M} \\ \nabla \times \mathbf{H}_{pF} = (\sigma + j\omega\varepsilon)\mathbf{E}_{pF} \end{cases}$$

$\mathbf{E}_{pA}, \mathbf{H}_{pA}$: fields due to electric source J

$$\begin{cases} -\nabla\times\mathbf{E}_{pA} = j\omega\mu\mathbf{H}_{pA} & \cdots \text{①} \\ \nabla\times\mathbf{H}_{pA} = (\sigma + j\omega\varepsilon)\mathbf{E}_{pA} + \mathbf{J} & \cdots \text{②} \end{cases}$$

$\nabla\cdot$① $\quad \rightarrow \ \nabla\cdot\mathbf{H}_{pA} = 0 \quad (\because \nabla\cdot\nabla\times\mathbf{a} = 0$: vector identity)

$$\mathbf{H}_{pA} = \nabla\times\mathbf{A}_p \qquad \cdots \text{③} \quad (\mathbf{A}_p \text{ : magnetic vector potential)}$$

③ $\rightarrow$ ① $\qquad \nabla\times(\mathbf{E}_{pA} + j\omega\mu\mathbf{A}_p) = \mathbf{0}$

$$\mathbf{E}_{pA} + j\omega\mu\mathbf{A}_p = -\nabla\phi_A \quad \cdots \text{④} \quad (\because \nabla\times\nabla\phi = \mathbf{0} \text{ : vector identity)}$$

③,④ $\rightarrow$ ②

$$\nabla\times\nabla\times\mathbf{A}_p - k^2\mathbf{A}_p = \mathbf{J} - (\sigma + j\omega\varepsilon)\nabla\phi_A$$

$$k^2 = -j\omega\mu(\sigma + j\omega\varepsilon)$$

$$\nabla(\nabla\cdot\mathbf{A}_p) - \nabla^2\mathbf{A}_p - k^2\mathbf{A}_p = \mathbf{J} - (\sigma + j\omega\varepsilon)\nabla\phi_A$$

$$(\because \nabla\times\nabla\times = \nabla\nabla\cdot - \nabla^2 \text{ : vector identity)}$$

$$\begin{cases} \nabla\times\mathbf{A}_p \text{ : } \textit{specified in } \text{③} \\ \nabla\cdot\ \mathbf{A}_p \text{ : } \textit{still free to be chosen} \end{cases}$$

$\nabla\cdot\boldsymbol{A}_p = -(\sigma + j\omega\varepsilon)\phi$: Lorentz condition (gauge)

$\underline{\nabla^2\mathbf{A}_p + k^2\mathbf{A}_p = -\mathbf{J}}$: vector Helmholtz Equation

$$\mathbf{E}_{pA} = -j\omega\mu\mathbf{A}_p + \frac{1}{\sigma + j\omega\varepsilon}\nabla(\nabla\cdot\mathbf{A}_p)$$

$$\mathbf{H}_{pA} = \nabla\times\mathbf{A}_p$$

Particular solutions

$$\mathbf{E}_p = \mathbf{E}_{pA} + \mathbf{E}_{pF} \quad , \quad \mathbf{H}_p = \mathbf{H}_{pA} + \mathbf{H}_{pF}$$

Maxwell Equation

$$\begin{cases} -\nabla \times \mathbf{E}_{pA} = j\omega\mu \mathbf{H}_{pA} \\ \nabla \times \mathbf{H}_{pA} = (\sigma + j\omega\varepsilon)\mathbf{E}_{pA} + \mathbf{J} \end{cases}$$

Electric source **J**

$$\begin{cases} -\nabla \times \mathbf{E}_{pF} = j\omega\mu \mathbf{H}_{pF} + \mathbf{M} \\ \nabla \times \mathbf{H}_{pF} = (\sigma + j\omega\varepsilon)\mathbf{E}_{pF} \end{cases}$$

Magnetic source **M**

Helmholtz Equation

$$\nabla^2 \mathbf{A}_p + k^2 \mathbf{A}_p = -\mathbf{J}$$

Electric source **J**

$$\nabla^2 \mathbf{F}_p + k^2 \mathbf{F}_p = -\mathbf{M}$$

Magnetic source **M**

Solution

$$\begin{cases} \mathbf{E}_{pA} = -j\omega\mu \mathbf{A}_p + \dfrac{1}{\sigma + j\omega\varepsilon} \nabla(\nabla \cdot \mathbf{A}_p) \\ \mathbf{H}_{pA} = \nabla \times \mathbf{A}_p \end{cases}$$

Electric source **J**

$$\begin{cases} \mathbf{E}_{pF} = -\nabla \times \mathbf{F}_p \\ \mathbf{H}_{pF} = -(\sigma + j\omega\varepsilon)\mathbf{F}_p + \dfrac{1}{j\omega\mu} \nabla(\nabla \cdot \mathbf{F}_p) \end{cases}$$

Magnetic source **M**

Complementary solutions (source-free： $\mathbf{J}=\mathbf{M}=\mathbf{0}$)

$$\mathbf{E}_c=\mathbf{E}_{cA}+\mathbf{E}_{cF}\quad,\quad \mathbf{H}_c=\mathbf{H}_{cA}+\mathbf{H}_{cF}$$

Maxwell Equation

$$\begin{cases}-\nabla\times\mathbf{E}_{cA}=j\omega\mu\mathbf{H}_{cA}\\ \nabla\times\mathbf{H}_{cA}=(\sigma+j\omega\varepsilon)\mathbf{E}_{cA}\end{cases}$$

$$\begin{cases}-\nabla\times\mathbf{E}_{cF}=j\omega\mu\mathbf{H}_{cF}\\ \nabla\times\mathbf{H}_{cF}=(\sigma+j\omega\varepsilon)\mathbf{E}_{cF}\end{cases}$$

Helmholtz Equation

$$\nabla^2\mathbf{A}_c+k^2\mathbf{A}_c=\mathbf{0}$$

$$\nabla^2\mathbf{F}_c+k^2\mathbf{F}_c=\mathbf{0}$$

Solution

$$\begin{cases}\mathbf{E}_{cA}=-j\omega\mu\mathbf{A}_c+\dfrac{1}{\sigma+j\omega\varepsilon}\nabla(\nabla\cdot\mathbf{A}_c)\\ \mathbf{H}_{cA}=\nabla\times\mathbf{A}_c\end{cases}$$

$$\begin{cases}\mathbf{E}_{cF}=-\nabla\times\mathbf{F}_c\\ \mathbf{H}_{cF}=-(\sigma+j\omega\varepsilon)\mathbf{F}_c+\dfrac{1}{j\omega\mu}\nabla(\nabla\cdot\mathbf{F}_c)\end{cases}$$

Fields in rectangular coordinates

$$\mathbf{A}_c = \hat{\mathbf{z}}\Psi_A \qquad \mathbf{F}_c = \hat{\mathbf{z}}\Psi_F$$

$$E_x = \frac{1}{\sigma + j\omega\varepsilon}\frac{\partial^2\Psi_A}{\partial x\partial z} - \frac{\partial\Psi_F}{\partial y} \qquad H_x = \frac{\partial\Psi_A}{\partial y} + \frac{1}{j\omega\mu}\frac{\partial^2\Psi_F}{\partial x\partial z}$$

$$E_y = \frac{1}{\sigma + j\omega\varepsilon}\frac{\partial^2\Psi_A}{\partial y\partial z} + \frac{\partial\Psi_F}{\partial x} \qquad H_y = -\frac{\partial\Psi_A}{\partial x} + \frac{1}{j\omega\mu}\frac{\partial^2\Psi_F}{\partial y\partial z}$$

$$E_z = \frac{1}{\sigma + j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\Psi_A \qquad H_z = \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\Psi_F$$

TM (transverse magnetic) to z

TE (transverse electric) to z

Arbitrary field $\cdots$ sum of a TM field and a TE field

Helmholtz Equation in rectangular coordinates

$$\mathbf{A}_c = \hat{\mathbf{z}}\Psi_A(x,y,z) \ \rightarrow \ \nabla^2\mathbf{A}_c + k^2\mathbf{A}_c = \mathbf{0}$$

$$\nabla^2\Psi_A(x,y,z) + k^2\Psi_A(x,y,z) = 0$$

$$\frac{\partial^2}{\partial x^2}\Psi_A(x,y,z) + \frac{\partial^2}{\partial y^2}\Psi_A(x,y,z) + \frac{\partial^2}{\partial z^2}\Psi_A(x,y,z) + k^2\Psi_A(x,y,z) = 0$$

$\Psi_A(x,y,z) = X(x)Y(y)Z(z)$ ：separation of variables

$$\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{-k_x^2} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{-k_y^2} + \underbrace{\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}}_{-k_z^2} + k^2 = 0$$

：each term is independent of x , y and z

$$\begin{cases} \dfrac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0 \\ \dfrac{d^2Y(y)}{dy^2} + k_y^2 Y(y) = 0 \\ \dfrac{d^2Z(z)}{dz^2} + k_z^2 Z(z) = 0 \end{cases} \rightarrow \begin{cases} X(x) = A_x e^{-jk_x x} + B_x e^{+jk_x x} \\ Y(y) = A_y e^{-jk_y y} + B_y e^{+jk_y y} \\ Z(z) = A_z e^{-jk_z z} + B_z e^{+jk_z z} \end{cases}$$

harmonic equations harmonic functions

$k_x^2 + k_y^2 + k_z^2 = k^2$ ：Only two of the k_i are chosen independently

harmonic functions

$$h(k_x x) \sim \sin k_x x\ ,\ \cos k_x x\ ,\ e^{+jk_x x}\ ,\ e^{-jk_x x}$$

Two of there are linearly independent

$$\Psi_{k_x k_y k_z} = \underbrace{h(k_x x)}_{X(x)}\ \underbrace{h(k_y y)}_{Y(y)}\ \underbrace{h(k_z z)}_{Z(z)}$$

Linear combination of wave functions

$$\Psi(x,y,z) = \sum_{k_x} \sum_{k_y} \underbrace{B_{k_x k_y}}_{\text{constant}} \Psi_{k_x k_y k_z} = \sum_{k_x} \sum_{k_y} B_{k_x k_y}\ h(k_x x)\ h(k_y y)\ h(k_z z)$$

k_x, k_y ：discrete spectra

determined by the boundary conditions of the problem

General wave function

$$\Psi(x,y,z) = \underbrace{\int_{k_x} \int_{k_y}}_{\text{any path in the complex } k_x \text{ and } k_y \text{ plane}} \underbrace{f(k_x, k_y)}_{\text{analytic function}} \Psi_{k_x k_y k_z}\ dk_x dk_y$$

k_x, k_y ：continuous spectrum

unbounded region