# Urban Environmental Engineering 3

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## Lecture in the Last time

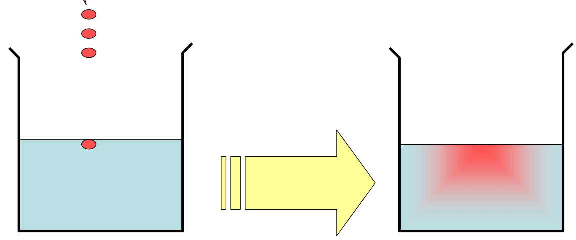
- Phenomena taking place in water environment
- Water quality parameters for Organic pollution
  - BOD, COD, TOC
- Streeter Phelps's equation for Change in BOD and DO. > We will discuss this issue in the lecture on reaction kinetics later.

## Today's Lecture

- Diffusion
  - Molecular diffusion
  - Turbulent diffusion
  - Dispersion

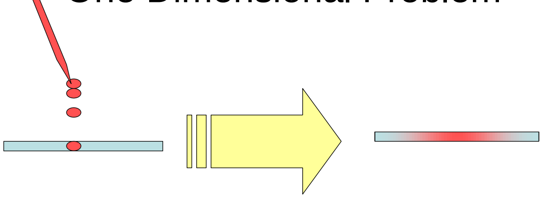
## One drop of red ink

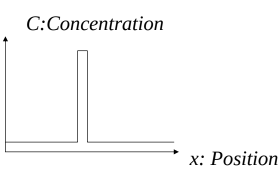
What is diffusion ?

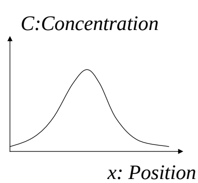


Time goes on.

## One Dimensional Problem





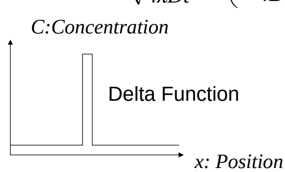

## If we write mathematic expressions,

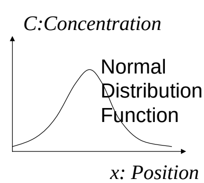
$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$ Equation of Change

$$C(x,0) = \delta(x)$$ Initial Condition

$$C(-\infty,t) = 0 \quad C(\infty,t) = 0$$ Boundary Condition

$$C(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$ Solution



# Exercises

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$ Equation of Change (1)

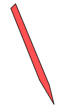
$$C(x,0) = \delta(x)$$ Initial Condition (2)

$$C(-\infty,t) = 0 \quad C(\infty,t) = 0$$ Boundary Condition (3)

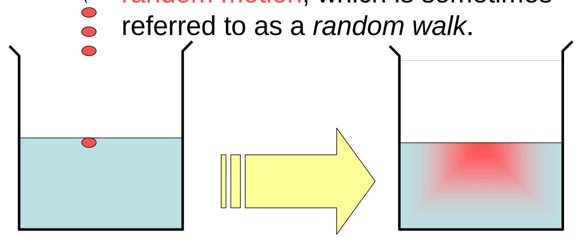
$$C(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$ Solution (4)

Q1: Show that equation (4) is really a solution of equation (1) under the initial condition (2) and the boundary condition (3). Draw the solution graphically with different time elapsed.

# Diffusion



Diffusion occurs as a result of random motion, which is sometimes referred to as a *random walk*.



Time goes on.

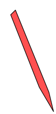
# Diffusion

If the random motion is caused by thermodynamic position exchange of molecules, the process is called molecular diffusion.

If the random motion is caused by turbulence, the process is called eddy diffusion or turbulent diffusion.

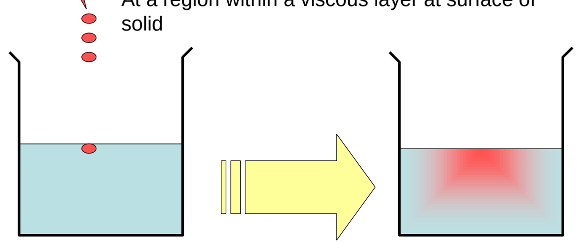
If the random motion is caused by different flow paths or different velocities in the field, the process is called dispersion.

# Molecular diffusion



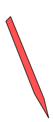
Water in a cup without motion of liquid.

At a region within a viscous layer at surface of solid



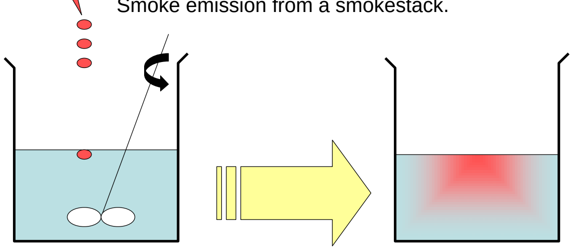
Time goes on.

# Turbulent diffusion



Water in a cup with stirrer.

Smoke emission from a smokestack.

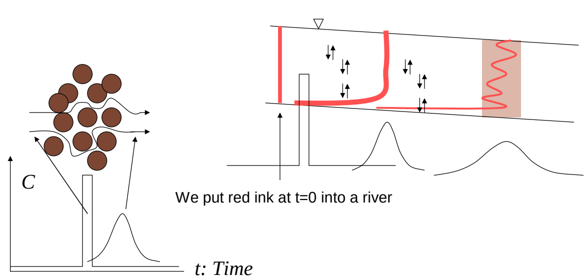


Time goes on.

# Dispersion

Contaminant transport in soil

Contaminant transport in river



# Mass Balance (No Flow, 1-D)

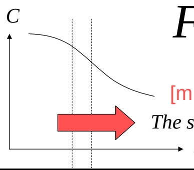
Heat is transferred from a high temperature side to a low temperature side. In the same way, mass is transferred from a high concentration side to a low concentration side.

The quantity of the transfer is in proportional to the gradient of concentration.

We name "Diffusivity" or " Coefficient of Diffusion"

$$Flux = -D\frac{\partial C}{\partial x}$$

[m2/s] in the SI system




*The solute will be transferred this direction*

# Analogy to momentum and heat transfer

$$MomentumFlux = -\nu\frac{\partial u}{\partial y}$$

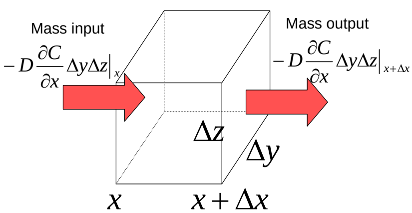
$$HeatFlux = -k\frac{\partial T}{\partial x}$$

Equation of change $\frac{\partial T}{\partial t} = \frac{k}{\rho C_P}\frac{\partial^2 T}{\partial x^2}$

$$MassFlux = -D\frac{\partial C}{\partial x}$$

Equation of change $\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$

# Mass Balance (No Flow, 1-D)

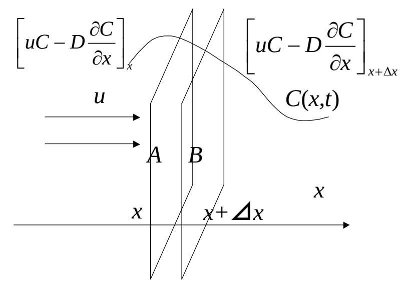



(Mass input) – (Mass Output) = (Accumulation)

$$-D\frac{\partial C}{\partial x}\Delta y\Delta z\Big|_x - \left(-D\frac{\partial C}{\partial x}\Delta y\Delta z\Big|_{x+\Delta x}\right) = \frac{\partial C}{\partial t}\Delta x\Delta y\Delta z$$

By dividing $\Delta x\Delta y\Delta z$ And taking limit of $x$ to 0 $\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$

# Mass Balance (with flow, 1-D)




Q2: Obtain a partial differential equation of one dimensional diffusion with flow.

# Extension to 3-D diffusion

If we can assume that the diffusion coefficient is a constant and the process shows no dependence on direction,

$$\frac{\partial C}{\partial t} + \frac{\partial uC}{\partial x} + \frac{\partial vC}{\partial y} + \frac{\partial wC}{\partial z} = D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)C$$

The general form of the transport equation is:

$$\frac{\partial C}{\partial t} + \frac{\partial uC}{\partial x} + \frac{\partial vC}{\partial y} + \frac{\partial wC}{\partial z} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}D_{xx} & D_{xy} & D_{xz}\\ D_{yx} & D_{yy} & D_{yz}\\ D_{zx} & D_{zy} & D_{zz}\end{pmatrix}\begin{pmatrix}\frac{\partial C}{\partial x}\\ \frac{\partial C}{\partial y}\\ \frac{\partial C}{\partial z}\end{pmatrix}$$

# Terms in the diffusion equation

$$\frac{\partial C}{\partial t} + \frac{\partial uC}{\partial x} + \frac{\partial vC}{\partial y} + \frac{\partial wC}{\partial z} = D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)C$$

Convection or advection

Diffusion

Time derivative(Local change of concentration)

## The comparison of diffusion equation to Navier Stokes equation

The diffusion equation obtained in the last slide has a similar terms in the Navier Stokes equation

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}+\frac{\partial vC}{\partial y}+\frac{\partial wC}{\partial z}=D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$

Time derivative

Convective term

Diffusion term
(Viscous term)

Same dimension
[m$^2$/s]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial w}{\partial z}=\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u$$

## Diffusion equation for non compressive flow

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}+\frac{\partial vC}{\partial y}+\frac{\partial wC}{\partial z}=D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$

Q3: Show that the above equation can be reduced to the following equation in the case of non compressive flow.

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$

## Extension to turbulent diffusion

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}+\frac{\partial vC}{\partial y}+\frac{\partial wC}{\partial z}=D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$

Substituting $C=\overline{C}+C', u=\overline{u}+u'$ and so on,

What will happen to the original equation ?

Q4: Show that the above equation can be extended to the following equation in the case of turbulent diffusion.

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}+\frac{\partial vC}{\partial y}+\frac{\partial wC}{\partial z}=(D+K)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$