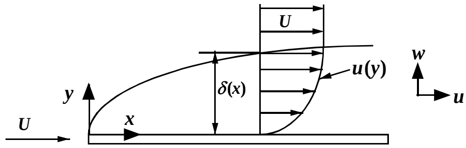
運動方程式, 連続の式



二次元定常

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$

→ **種々仮定により簡略化**

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}=0$$

無次元化: $u^*=\frac{u}{U}, w^*=\frac{w}{W}, x^*=\frac{x}{L}, y^*=\frac{y}{\delta_0}, p^*=\frac{p}{\rho U^2}$

各項のオーダー

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

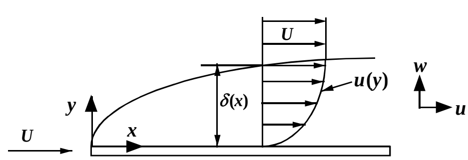
右辺

$$\nu\frac{\partial^2 u}{\partial x^2}=\nu\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}\rightarrow\nu\frac{U}{L^2}$$

$$\nu\frac{\partial^2 u}{\partial y^2}=\nu\frac{U}{{\delta_0}^2}\frac{\partial^2 u^*}{\partial x^{*2}}=\nu\frac{U}{L^2}\left(\frac{L^2}{{\delta_0}^2}\right)\frac{\partial^2 u^*}{\partial x^{*2}}\rightarrow\nu\frac{U}{L^2}\left(\frac{L^2}{{\delta_0}^2}\right)$$

↓ $\frac{\delta_0}{L}<<1$

$$\frac{\partial^2 u}{\partial x^2}<<\frac{\partial^2 u}{\partial y^2}$$



$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

右辺つづき

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{1}{\rho}\frac{\rho U^2}{L}\frac{\partial p*}{\partial x*}\rightarrow\frac{U^2}{L}$$

左辺

$$u\frac{\partial u}{\partial x}=\frac{U^2}{L}u*\frac{\partial u*}{\partial x*}\rightarrow\frac{U^2}{L}$$

$$w\frac{\partial u}{\partial y}=\frac{WU}{\delta_0}w*\frac{\partial u*}{\partial x*}\rightarrow\frac{WU}{\delta_0}$$

Wのオーダー？･･･保留

移動現象操作

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}\right)$$

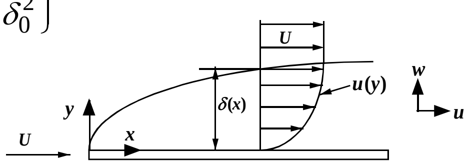
左辺と右辺の比較

$$\left.\begin{aligned}u\frac{\partial u}{\partial x}&\rightarrow\frac{U^2}{L}\\ \frac{1}{\rho}\frac{\partial p}{\partial x}&\rightarrow\frac{U^2}{L}\\ \nu\frac{\partial^2 u}{\partial y^2}&\rightarrow\nu\frac{U}{\delta_0^2}\end{aligned}\right\}$$

同オーダー →

$$\frac{U^2}{L}=\nu\frac{U}{\delta_0^2}$$

$$\delta_0=\sqrt{\frac{\nu L}{U}}=L\sqrt{\frac{\nu}{UL}}\rightarrow L\,\mathrm{Re}^{-1/2}$$



$$\delta_0 = \sqrt{\frac{\nu L}{U}} = L\sqrt{\frac{\nu}{UL}} \rightarrow L\,\mathrm{Re}^{-1/2}$$

↓

保留したWのオーダーを考える

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} = 0$$ **連続の式**

↓

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{U}{L}\frac{\partial u^*}{\partial x^*} \rightarrow \frac{U}{L} \\ \frac{\partial w}{\partial y} &= \frac{W}{\delta_0}\frac{\partial w^*}{\partial y^*} \rightarrow \frac{W}{\delta_0} \end{aligned}\right\}$$

同オーダー ⟶ $W \rightarrow \frac{\delta_0 U}{L} = U\,\mathrm{Re}^{-1/2}$

移動現象操作

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$w\frac{\partial u}{\partial y} = \frac{WU}{\delta_0}w^*\frac{\partial u^*}{\partial y^*} \rightarrow \frac{U^2}{L} \qquad \left(W \rightarrow \frac{\delta_0 U}{L}\right)$$

$$\nu\frac{\partial^2 u}{\partial x^2} = \nu\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} \rightarrow \frac{U\delta_0^2}{L}\frac{U}{L^2}$$

その他の項 $\rightarrow \frac{U^2}{L}$

↓

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$

$$u\frac{\partial w}{\partial x}=\frac{UW}{L}u^*\frac{\partial w^*}{\partial x^*}\rightarrow\frac{UW}{L}=\frac{U}{L}\frac{\delta_0 U}{L}=\frac{U^2}{L^2}\delta_0$$

$$w\frac{\partial w}{\partial y}=\frac{W^2}{\delta_0}w^*\frac{\partial w^*}{\partial y^*}\rightarrow\frac{W^2}{\delta_0}=\frac{1}{\delta_0}\frac{\delta_0^2 U^2}{L^2}=\frac{U^2}{L^2}\delta_0$$

$$\nu\frac{\partial^2 w}{\partial x^2}=\frac{U\delta_0^2}{L}\frac{W}{L^2}\frac{\partial^2 w^*}{\partial x^{*2}}\rightarrow\frac{U\delta_0^2 W}{L^3}=\frac{U^2}{L^4}\delta_0^3$$

$$\nu\frac{\partial^2 w}{\partial y^2}=\frac{U\delta_0^2}{L}\frac{W}{\delta_0^2}\frac{\partial^2 w^*}{\partial y^{*2}}\rightarrow\frac{UW}{L}\rightarrow\frac{U^2}{L^2}\delta_0$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

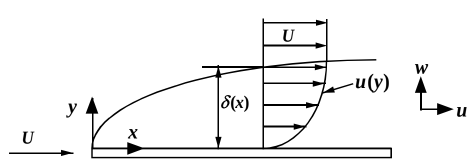
境界層方程式

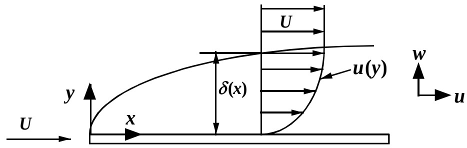
$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$ → **境界層内圧力＝主流圧力**

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}=0$$

境界条件: $u=w=0$ at $y=0$, $u=U$ at $y=\infty$





主流の圧力は？

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

↓ $u=U,\ w=0$ **粘性項**=0

$$U\frac{\partial U}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

↓ **積分**

$$\frac{1}{2}U^2=-\frac{p}{\rho}+\mathrm{Const.} \longrightarrow \frac{1}{2}U^2+\frac{p}{\rho}=\mathrm{Const.}$$

再び境界層内の方程式

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

↓ $U\frac{\partial U}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}=0$ （Uは流れ方向に変化しない）

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

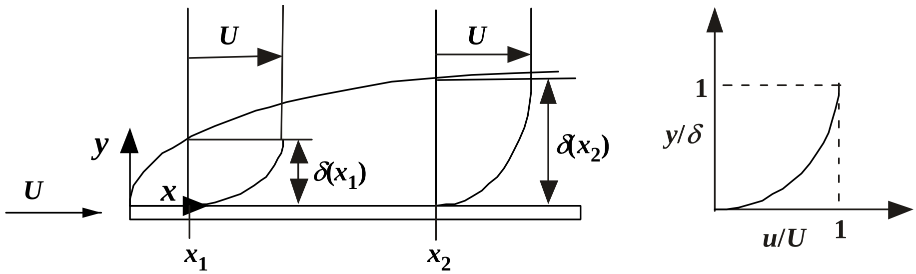
独立変数2
従属変数2
偏微分方程式

⟶

独立変数1
従属変数1
常微分方程式

- **速度分布の相似**
- **流れ関数の導入**

速度分布の相似



$$\delta \propto \delta_0 = \sqrt{\frac{\nu x}{U}} \rightarrow \eta \equiv \frac{y}{\sqrt{\frac{\nu x}{U}}} = \frac{y}{x}\mathrm{Re}^{1/2} \qquad \left(\mathrm{Re} = \frac{Ux}{\nu}\right)$$

$$\frac{u}{U} = g(\eta)$$

流れ関数

$$\frac{dy}{dx} = \frac{w}{u} \rightarrow \frac{dx}{u} = \frac{dy}{w} \rightarrow udy - wdx = 0$$

非圧縮性流体 $\frac{\partial u}{dx} + \frac{\partial w}{dy} = 0$

↓

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -wdx + udy = 0$$

$$u = \frac{\partial \psi}{\partial y},\ w = -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \psi}{\partial y}$$

↓ yで積分

$$\psi = \int u dy + h(x)$$

$$\frac{u}{U} = g(\eta) \to u dy = U g(\eta) \frac{\partial y}{\partial \eta} d\eta$$

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U}}} \to \frac{\partial y}{\partial \eta} = \sqrt{\frac{\nu x}{U}}$$

↓

$$\psi = U\sqrt{\frac{\nu x}{U}} \int g(\eta) d\eta + h(x) = \sqrt{\nu x U} \int g(\eta) d\eta + h(x)$$

$h(x)$は？

$$w = -\frac{\partial \psi}{\partial x} = \sqrt{\nu x U} g(\eta) \frac{\partial \eta}{\partial x} + \frac{1}{2} \int g(\eta) d\eta \sqrt{\frac{\nu U}{x}} + \frac{\partial h(x)}{\partial x}$$

↓ $w = 0$ at $y = 0$

$$\frac{\partial h(x)}{\partial x} = 0 \to h(x) = \text{Const.}$$

$$\psi = U\sqrt{\frac{\nu x}{U}} \int g(\eta) d\eta + h(x) = \sqrt{\nu x U} \int g(\eta) d\eta + h(x)$$

$$\sqrt{\nu x U} \int g(\eta) d\eta = 0 \to \psi = 0 \to h(x) = 0$$

$$h(x) = 0, \quad \int g(\eta) d\eta = f(\eta)$$

とすると

↓

$$\psi = \sqrt{\nu x U} \int g(\eta) d\eta = \sqrt{\nu x U} f(\eta)$$

$$\psi = \sqrt{\nu x U}\int g(\eta)d\eta = \sqrt{\nu x U}\, f(\eta)$$

$$u = \frac{\partial \psi}{\partial y} = \sqrt{\nu x U}\,\frac{\partial f(\eta)}{\partial y} = \sqrt{\nu x U}\,\frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} = Uf'(\eta)$$

$$w = -\frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\{\eta f'(\eta) - f(\eta)\}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = -Uf''(\eta)\frac{\eta}{2x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = Uf''(\eta)\sqrt{\frac{U}{\nu x}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\nu x}\, f'''(\eta)$$

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \longrightarrow 2f''' + ff'' = 0$$

$$2f''' + ff'' = 0$$

$$u = Uf'(\eta) = 0 \text{ at } \eta = 0 \rightarrow f'(\eta) = 0$$

$$w = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\{\eta f'(\eta) - f(\eta)\} = 0 \text{ at } \eta = 0 \rightarrow f(\eta) = 0$$

$$u = Uf'(\eta) = U \text{ at } \eta = \infty \rightarrow f'(\eta) = 1$$

$f(\eta)$:**ブラジウス**(Blasius)**の近似解**

- **平板近傍** $\eta \approx 0$ **級数に展開して代入**
- **境界層端付近** $\eta \rightarrow \infty$ **近似解に微小の補正を加えた関数**

↓

二つの解がつながるように定数を決定

Howarth**の解**

境界層厚さの定義

平板から流速が主流の99%となる位置まで $\frac{u}{U} = 0.99$

↑

実際に厚さを決定するのは困難

1.排除厚さ

平板があることにより失われた流量

$$\int_0^\infty (U - u)dy$$

排除厚さ：δ^* 失われた流量が占めて いたはずの層の厚さ

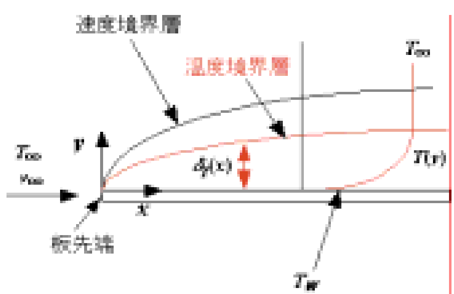
$$U\delta^* = \int_0^\infty (U - u)dy \rightarrow \delta^* = \int_0^\infty (1 - \frac{u}{U})dy$$

2.運動量厚さ

平板があることにより失われた運動量 $\int_0^\infty \rho u(U - u)dy$

運動量厚さ：θ $\rho U^2\theta = \rho\int_0^\infty u(U - u)dy \rightarrow \theta = \int_0^\infty \frac{u}{U}(1 - \frac{u}{U})dy$

熱移動方程式

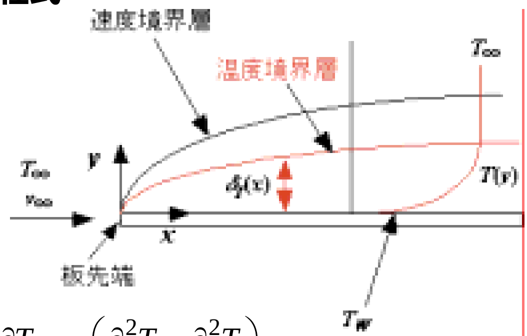


$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

↓

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

熱移動方程式



$$u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

$$\downarrow \quad \theta=\frac{T-T_w}{T_\infty-T_w}\rightarrow\frac{\partial\theta}{\partial x}=\frac{1}{T_\infty-T_w}\left(\frac{\partial T}{\partial x}-\frac{\partial T_w}{\partial x}\right)=\frac{1}{T_\infty-T_w}\frac{\partial T}{\partial x}$$

$$u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial y}=\alpha\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right)$$

$$u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial y}=\alpha\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right)$$

$$\downarrow$$

$$u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial y}=\alpha\frac{\partial^2 \theta}{\partial y^2}$$

$$\psi=\sqrt{\nu xU}\int g(\eta)d\eta=\sqrt{\nu xU}\,f(\eta)$$

$$u=\frac{\partial\psi}{\partial y}=\sqrt{\nu xU}\frac{\partial f(\eta)}{\partial y}=\sqrt{\nu xU}\frac{\partial f(\eta)}{\partial\eta}\frac{\partial\eta}{\partial y}=Uf'(\eta)$$

$$\downarrow \quad w=-\frac{\partial\psi}{\partial x}=-\frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x}=\frac{1}{2}\sqrt{\frac{\nu U}{x}}\{\eta f'(\eta)-f(\eta)\}$$

$$\eta=\frac{y}{\sqrt{\frac{\nu x}{U}}}$$

$$2\frac{\partial^2\theta}{\partial\eta^2}+\mathrm{Pr}\ f\frac{\partial\theta}{\partial\eta}=0$$

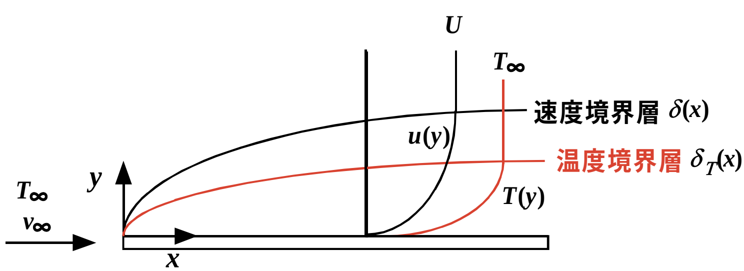
$$2f'''+ff''=0$$

速度境界層の式との比較

$$\frac{u}{U}=g(\eta)=f'$$

$$\frac{\partial g}{\partial\eta}=f'',\ \frac{\partial^2 g}{\partial^2\eta}=f'''$$

$$2\frac{\partial^2 g}{\partial\eta^2}+f\frac{\partial g}{\partial\eta}=0$$



$$\mathrm{Pr}\equiv\frac{\nu}{\alpha}$$